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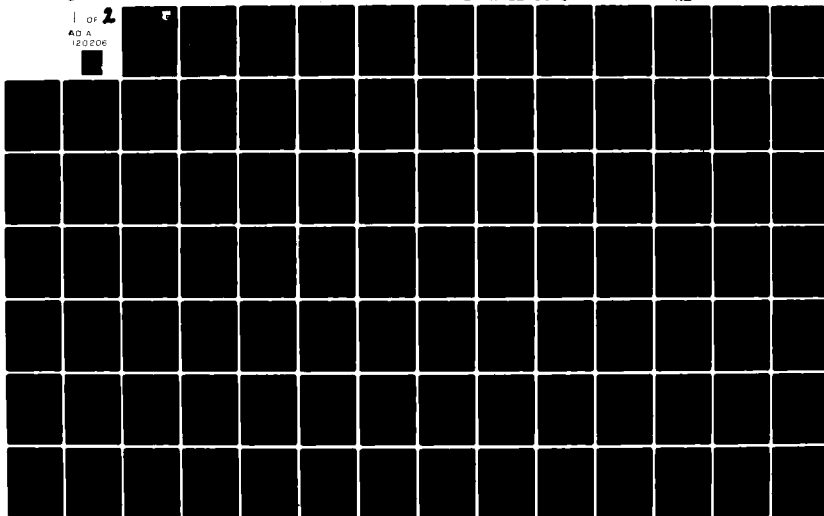
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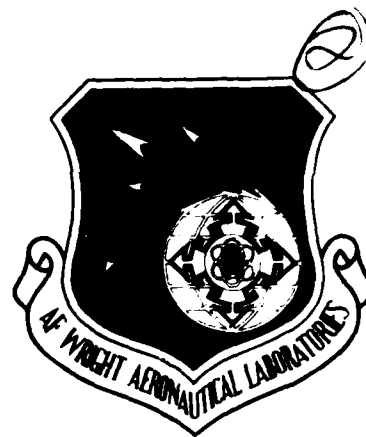
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THE INTEGRAL EQUATION METHOD FOR TRANSONIC FLOW
INTERPRETED AS METHOD OF WEIGHTED RESIDUALS



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September 1982
Interim Report for Period January 1980 - August 1981

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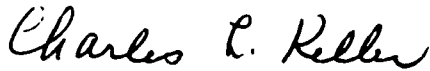
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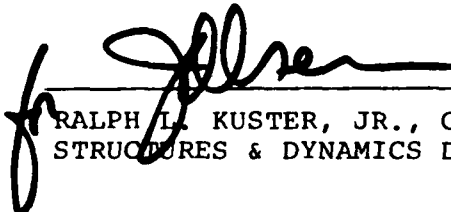


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latter methods, because of their strongly-localized character are particularly well-suited to treat the transition through the sonic line and shocks. The integral equation method is best in the subsonic part of the flow field. Using the integral equation method only in the far field, one obtains far field conditions which approximately take into account nonlinear terms even in the far field, and, therefore, are more accurate than far field conditions so far available in the literature.

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PREFACE

This report has been written under Contract F33615-81-K-3216 entitled, "Mathematical Questions Related to the Computation of Compressible Flow Field," to the University of Dayton for the Applied Mathematics Group, Analysis and Optimization Branch, Structures and Dynamics Division, AFWAL/FIBRC under Project 2304, Task 2304N1, and Program Element 61102F. The work was performed during the period June through December 1981. Dr. Karl G. Guderley of the University of Dayton Research Institute was Principal Investigator. Dr. Charles L. Keller, AFWAL/FIBRC, (513) 255-7384, Wright-Patterson Air Force Base, Ohio 45433 was program manager.

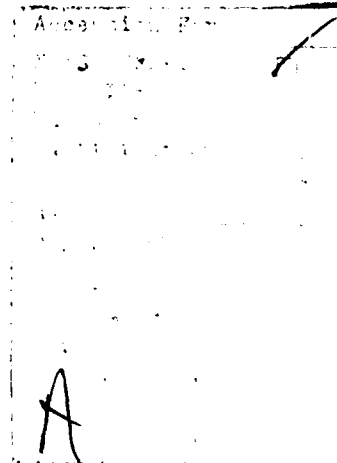


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SECTION I

INTRODUCTION

During the last few years numerical methods for predicting the flow field around an airfoil have advanced tremendously. The computation effort is, however, quite large; there may, therefore, still be room for methods which yield results with acceptable accuracy but with less computational work. This claim is made for the integral equation method, originally proposed by Oswatitsch (Ref. 1) and developed further by Spreitzer, Zierep, Norstrud, Hancock, and Nixon (Ref. 2 through 9). The claim to shorter computing time is made on the basis of numerical experiments. The following observations, which probably have already been in the minds of the originators can serve as an explanation. The integral equation formulation contains terms which are found directly from the Prandtl Glauert approximation and additional terms which originate from further (nonlinear) compressibility effects. Frequently these additional terms are fairly small; it may be permissible to evaluate them with relaxed accuracy requirements.

The present article analyzes the method from a theoretical point of view. The author's original goal was to identify the feature of the method which is the basis for its success. The explanation which he found is nothing more than a paraphrase in mathematical terms of the reason just given; it has little practical interest. Further developments of the underlying ideas, namely the interpretation of the integral equation method as a method of weighted residuals, may however lead to improvements of the computational procedure and this should make the present study worthwhile.

SECTION II

GENERAL OBSERVATIONS AND A SURVEY OF THE ANALYSIS

The discussion is based on two fairly simply observations. The first is the basic motivation for the definition of a Green's function: the Green's function pertaining to some linear differential operation (with appropriate homogeneous boundary conditions) constitutes the kernel of an integral operator which inverts the original differential operator. If one uses, instead of the Green's function a fundamental solution, then one obtains only a partial inversion of the differential operator, but one still accomplishes a reduction of the dimensionality of the problem by one.

With the exact Green's function one knows, of course, the solution of a given problem except for some quadratures. Let us assume nevertheless that one carries out a method of weighted residuals with the Green's function as weight function. Accordingly, one assumes some form of the approximate solution which satisfies the boundary conditions. It will depend upon a number (usually very large) of unknown parameters. One then expresses the residuals within the flow field in terms of these parameters and forms the integral over the residuals weighted with the Green's function for which the singular point lies at a certain value of (x,y) . Applying Green's formula to the integral over the whole field with the point (x,y) excluded by a small circle, one finds that the solution at the chosen point (x,y) is expressed solely in terms of the prescribed boundary values. One thus finds the exact solution for the point (x,y) in spite of the fact that the residuals in the field do not vanish. The Green's function with a singular point (x,y) therefore constitutes a weight function which makes the solution at the point (x,y) insensitive to residuals within the flow field. Incidentally, in a nonlinear problem one must consider the differential operator which arises by linearizing the original partial differential equation for the vicinity of a certain approximation, the so called Frechet derivative. The

Green's function pertaining to the Frechet derivative makes the result for a certain point (x,y) insensitive against first order terms in the residual.

If only an approximation to the Green's function is available, then the residuals within the flow field will give a contribution to the expression for the solution at the point (x,y) . This contribution is, of course, small if the approximation to the Green's function is close and then it is permissible to evaluate the effect of the residuals with reduced accuracy. Actually, this interpretation is not much more than the argument put forth above. An approximate Green's function is available only in special cases (mainly flows which are symmetric with respect to the x axis). Further work, namely the solution of an integral equation is necessary, if only an approximate fundamental solution is available, but the main effect of the use of such weight functions is the same.

The present report studies a second observation in considerable detail. We shall interpret the integral equation approach to transonic flow as a method of weighted residuals in which the weight functions are given by approximate fundamental solutions (and in favorable cases by approximate Green's functions). Each choice of the singular point (x,y) in the Green's function gives one weight function. In methods of weighted residuals one has considerable freedom in the choice of weight functions. One can therefore replace some of the weight functions which are used in the integral equation method, by weight functions of a different character. This makes it possible to combine the integral equation method with finite element or finite difference approaches. This is advisable in regions where the available approximation to the ideal fundamental solution is poor. One thus is led to alternative versions of Nixon's extended integral equation method and, in a further extension, to an improvement in the far field conditions for the lower transonic range. This last development ought to prove useful even in conjunction with conventional finite differences approaches, for

it makes possible a reduction of the size of the portion of the flow field in which the computation is carried out.

Section III of the report is a compilation of different formulations of the problem in terms of differential equations. Section IV is an overview of different integral equation formulations. This seems to be desirable not only because we look at the method in a different light, but also because most publications treat the theoretical side rather briefly since they are mainly oriented towards the numerical aspects. One question is particularly intriguing: does the integral equation method fully express the shock conditions, or it is necessary to express part of the shock conditions separately. Here physical intuition cannot be considered as a reliable guide because of the global character of the integral equation formulation. One may, of course, resort to numerical experimentation, and this has been done in the past. But even for those who are primarily oriented toward the computational practice, the present analysis may provide reassuring background information.

Only the important results are compiled in Section IV. Details are found in a number of appendices. To someone well versed in the mathematical techniques used in this context, a more condensed version would probably be sufficient, but, at least some of the details are desirable if one wants to show the subtle differences in the assumptions and in the arguments between different formulations.

Section V considers the conditions that are to be imposed at the transition from a subsonic to a supersonic flow at the sonic line. In principle the condition to be satisfied has local character and the global character of the integral equation formulation is not too well-suited for this purpose. An ingenious device has been introduced by Spreiter, which allows one to overcome this difficulty; its mathematical meaning will be explored. The modifications of the method which are discussed in the subsequent sections will probably make the use of this device unnecessary, but it is of interest to see why it works.

Section VI describes the original form of the integral equation method, the extension proposed by Nixon, and the modifications suggested by the present interpretation. Section VII examines the relation between the integral equation formulation and the far field conditions.

The present report does not claim to give a complete survey of the integral equation method. Finer points, the extension to three dimensional problems, and to problems in which small harmonic oscillations are superimposed to a steady field will not be treated. To carry over the ideas of the present report to problems of this kind is not difficult. In the practical realization of the modification suggested here, experiences gained in the past with the integral equation method ought to prove useful.

SECTION III

PARTIAL DIFFERENTIAL EQUATIONS AND SHOCK CONDITIONS

The investigation starts with the potential equation simplified for transonic flow:

$$(1 - M^2 - \kappa \bar{\phi}_x) \bar{\phi}_{xx} + \bar{\phi}_{yy} = 0 \quad (1)$$

where \bar{x} and \bar{y} are Cartesian coordinates, $\bar{\phi}$ the velocity potential, M the free stream Mach number, and κ a constant which arises by a development of $1 - (\bar{c}_x^2/\bar{a}^2)$ for the vicinity of the free stream Mach number; \bar{a} is the free stream sound velocity. For a free stream Mach number one, κ is equal to $(\gamma + 1)$ where γ is the ratio of the specific heats. The equation is brought into a standard form by means of the Prandtl Glauert coordinate distortion

$$\begin{aligned} y &= (1 - M^2)^{1/2} \bar{y} \\ x &= \bar{x} \end{aligned} \quad (2)$$

and by setting

$$\phi = \kappa (1 - M^2)^{-1} \bar{\phi}. \quad (3)$$

One obtains

$$(1 - \phi_x) \phi_{xx} + \phi_{yy} = 0 \quad (4)$$

or

$$\frac{\partial}{\partial x} \left(\phi_x - \frac{\phi_x^2}{2} \right) + \phi_{yy} = 0. \quad (5)$$

Eq. (4) shows that sonic speed occurs for $\phi_x = 1$.

In simplifying the shock conditions for transonic flow, one retains the requirement that the momentum equation for the direction tangential to the shock be satisfied. This requires that the potential be continuous as one traverses the shock.

The energy equation is always satisfied if one computes the pressures from the Bernoulli equation. In addition, the requirement of conservation of mass is retained. It finds its expression in the potential equation. If one postulates that across the shock, the potential equation is satisfied in the weak sense, then conservation of mass is guaranteed. This holds, of course, for Eq. (5) as well as for the original potential equation, Eq. (1). A fourth shock condition, preservation of momentum in the direction normal to the shock, must be waived if one works with a flow potential. The errors so introduced are of third order in the shock intensity (expressed, for instance, by the jump of the Mach number across the shock). This is in keeping with the approximation made in Eq. (1).

A method of weighted residuals amounts to the use of the concept of weak equality. The shock conditions, therefore, appear automatically, if one solves the potential equation by a method of weighted residuals. A derivation of the shock conditions using the concept of generalized functions (for which the concept of weak equality plays a fundamental role) is shown in Appendix 1. We quote the result in the following.

Let the shock be given by

$$x = x_s(y)$$

and let $[H]_-^+$ the jump of some quantity, H , across the shock from the upstream to the downstream side

$$[H]_-^+ = H(x_s(y) - \epsilon, y) - H(x_s(y) + \epsilon, y) \quad (6)$$

$$\epsilon > 0, \quad \epsilon \rightarrow 0.$$

Let β be the angle of the normal to the shock with the x -axis. One has

$$\frac{dx_s}{dy} = -\tan \beta \quad (7)$$

Conservation of mass and continuity of ϕ across the shock are expressed by

$$[\phi_x - \frac{\phi_x^2}{2}]_-^+ - [\phi_y]_-^+ \frac{dx_s}{dy} = 0 \quad (8)$$

and

$$[\phi]_-^+ = 0. \quad (9)$$

An equivalent formulation is obtained by introducing

$$\phi_x = u \quad (10)$$

$$\phi_y = v.$$

(The actual velocity components can be obtained from u and v by the transformations Eqs. (2) and (3).)

Then one has

$$u_y = v_x \quad (11)$$

and from Eq. (5)

$$\frac{\partial}{\partial x} (u - \frac{u^2}{2}) + \frac{\partial}{\partial y} v = 0. \quad (12)$$

The shock condition (8) then assumes the form

$$[u - (1/2)u^2]_-^+ - [v]_-^+ (dx_s/dy) = 0. \quad (13)$$

An expression equivalent to Eq. (9), but in terms of u and v is obtained by differentiating Eq. (9) along the shock

$$[u]_-^+ (dx_s/dy) + [v]_-^+ = 0. \quad (14)$$

One might be tempted to derive Eq. (14) from Eq. (11) by means of the concept of weak equality, but Eq. (11) is not the expression for the conservation of some physical quantity, although it has divergence form. Such a procedure would therefore not be physically justified, although it gives the correct result.

A formulation of the problem solely in terms of u , which has originally been used in the integral equation method, is obtained by differentiating Eq. (5) with respect to x . One obtains

$$\frac{\partial^2}{\partial x^2} \left(u - \frac{u^2}{2} \right) + \frac{\partial^2}{\partial y^2} u = 0. \quad (15)$$

In the approximation used in all of these derivations, the boundary conditions at the profile are transferred to the x axis. Along the upper and lower side of the profile the values of v are then given by the profile shape. In the formulation in terms of u the values of $u_y = v_x$ are known at the profile.

The shock conditions (13) are expressed solely in terms of u by using Eq. (14). Using, in addition, Eq. (7) one obtains

$$[u]_-^+ - \left[\frac{u^2}{2} \right]_-^+ \cos^2 \beta = 0. \quad (16)$$

To express Eq. (14) solely in terms of u , we differentiate it along the shock with respect to y . This means we apply the operator

$$\frac{\partial}{\partial y} + \frac{dx_s}{dy} \frac{\partial}{\partial x}$$

One obtains

$$\begin{aligned} \frac{\partial}{\partial y} [u]_-^+ \frac{dx_s}{dy} + \frac{\partial}{\partial x} [u]_-^+ \left(\frac{dx_s}{dy} \right)^2 + [u]_-^+ \frac{d^2 x_s}{dy^2} + \frac{\partial}{\partial y} [v]_-^+ \\ + \frac{\partial}{\partial x} [v]_-^+ \frac{dx_s}{dy} = 0 \end{aligned}$$

Here Eqs. (11) and (12) are used to eliminate v . One obtains

$$2 \frac{\partial}{\partial y} [u]_-^+ \frac{dx_s}{dy} + \frac{\partial}{\partial x} [u]_-^+ \left(\frac{dx_s}{dy} \right)^2 - \frac{\partial}{\partial x} \left[u - \frac{u^2}{2} \right]_-^+ + [u]_-^+ \frac{d^2 x_s}{dy^2} = 0. \quad (17)$$

Eq. (16) and (17) give the complete formulation of the shock conditions. In practice one will probably avoid the formulation solely in terms of u because of the complexity of the shock condition, (Eq. 17).

If a u -field satisfying Eq. (15) (but not necessarily Eqs. (16) and (17)) is known, then one can always construct a v -field by the use of Eq. (11) and Eq. (12). One determines, along some initial line $x = \text{const}$ which intersects all lines $y = \text{const}$, the velocity component v from Eq. (11) by an integration with respect to y . It is best to choose this line at $x = -\infty$. There, all derivatives of u are zero, and therefore $v = 0$. One then determines v by integrating Eq. (11) along lines $y = \text{const}$. This construction cannot reach points which lie downstream of the shock. For these points one must again use Eq. (12), along a line $x = +\infty$ to obtain initial conditions for v . Again, one obtains $v = 0$. (Actually some discussion of the behavior of u_y at large values of y is needed to arrive at this result.)

The differential equation in terms of u , Eq. (15), arises from the equation for conservation of mass by a differentiation with respect to x . Conservation of mass is therefore guaranteed only if conservation of mass is satisfied along some initial line. The construction of the v -field described here imposes this requirement and in this manner legitimizes the use of Eq. (15). Whenever a u -field has been computed, one can assume that the pertinent v -field is available.

SECTION IV

INTEGRAL EQUATION FORMULATIONS

In the following discussions a weight function denoted by ψ will be used. The following expression has been chosen:

$$\psi(\xi-x, \eta-y) = (1/2) \log((\xi-x)^2 + (\eta-y)^2). \quad (18)$$

The umbral variables operating in future integral expressions are the variables ξ and η which appear in this equation. They correspond to x and y . Each choice of the point (x,y) defines one weight function. The weight functions satisfy

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} = 0 \quad (19)$$

except at the point $(\xi, \eta) = (x, y)$. These weight functions may appear unconventional because they change sign at $(\xi-x)^2 + (\eta-y)^2 = 1$ and because they tend to infinity if the argument is zero or infinite. Actually, any weight function $w(\xi, \eta)$ can be written as

$$w(\xi, \eta) = \frac{1}{2\pi} \iint \Delta w(\xi_1, \eta_1) \psi(\xi_1 - \xi, \eta_1 - \eta) d\xi_1 d\eta_1 \quad (20)$$

where Δ denotes the Laplace operator and ξ_1 and η_1 are umbral variables of integration. This is seen by a standard application of Green's theorem to the equation

$$\Delta w(\xi_1, \eta_1) = f(\xi_1, \eta_1).$$

One forms

$$\iint \Delta w(\xi_1, \eta_1) \psi(\xi_1 - \xi, \eta_1 - \eta) d\xi_1 d\eta_1 = \iint f(\xi_1, \eta_1) \psi(\xi_1 - \xi, \eta_1 - \eta) d\xi_1 d\eta_1.$$

Assuming that w and its first derivative vanish outside of a certain region, one obtains for the left hand side $2\pi w(\xi, \eta)$.

This leads directly to Eq. (20). The set of weight functions ψ is therefore equivalent to conventional weight functions of finite support (such weight functions are different from zero only in a finite region).

We now substitute some function (again denoted by ϕ) into the potential Equation (5). If the differential equation is not satisfied, then one obtains a residual. We evaluate the integral over the residual weighted with the function ψ . The resulting expression is denoted by $Q(x,y)$. The arguments (x,y) in Q refer to the same arguments in ψ . The independent variables x and y in Eq. (5) are replaced by ξ and η . Accordingly, we form

$$Q(x,y) = \iint_{\Omega} \left[\frac{\partial}{\partial \xi} (\phi_{\xi} - \phi_{\xi}^2/2) + \frac{\partial}{\partial \eta} (\phi_{\eta}) \right] \psi(\xi-x, \eta-y) d\xi d\eta. \quad (21)$$

After certain transformations which are shown in detail in Appendix II, one obtains:

$$Q(x,y) = 2\pi\phi(x,y) - \int_0^L \psi(\xi-x, -y) (\phi_{\eta}(\xi, +0) - \phi_{\eta}(\xi, -0)) d\xi \\ + \int_0^{\infty} \psi_{\eta}(\xi-x, -y) (\phi(\xi, +0) - \phi(\xi, -0)) d\xi + (1/2) \iint_{\Omega} \psi_{\xi}(\xi-x, \eta-y) \phi_{\xi}^2(\xi, \eta) d\xi d\eta \quad (22)$$

Here, Ω denotes the entire ξ, η plane punctured at the point $(\xi, \eta) = (x, y)$. The profile extends along the x axis from $x = 0$ to $x = L$. It appears as a cut in the ξ, η plane. In the presence of a circulation (denoted by Γ) the cut is extended along the wake (again represented by the x axis). The arguments $+0$ and -0 refer to the limiting values assumed by η as one approaches the cut.

Eq. (22) does not contain explicit contributions due to the shock; the shock conditions are included by the use of the concept of weak equality which is inherent in a weighted residual formulation. A failure to satisfy the shock conditions registers in the expression

Q in the same manner as a failure to satisfy, in a certain region, the partial differential Eq. (5). If one chooses to exclude the shock from the region Ω by a cut, then the first expression is modified in the following manner. We denote by Ω' the region Ω cut along the shock and by Q' the pertinent function Q. The region Ω' has the two borders of the cut as part of its boundary. Then

$$\begin{aligned}
 Q'(x,y) = & Q(x,y) + \int \psi(\xi_S(\eta) - x, \eta - y) \{ [\phi_\xi(\xi_S(\eta), \eta)]_-^+ - \frac{1}{2} [\phi_\xi^2(\xi_S(\eta), \eta)]_-^+ \\
 & - [\phi_\eta(\xi_S(\eta), \eta)]_-^+ (d\xi_S/d\eta) \} d\eta \\
 & - \int [\phi(\xi_S(\eta), \eta)]_-^+ \{ \psi_\xi(\xi_S(\eta) - x, \eta - y) - \psi_\eta(\xi_S(\eta) - x, \eta - y) (d\xi_S/d\eta) \} d\eta.
 \end{aligned} \tag{23}$$

The integrals are extended along the shocks in the positive η -direction. The second integral vanishes if ϕ is continuous along the cut. The first integral vanishes if the shock conditions Eq. (8) are satisfied. In principle, continuity of the potential and conservation of mass are separate postulates, if the shock is excluded by a cut.

In a symmetric flow pattern, one has

$$\phi(\xi, +0) = \phi(\xi, -0) \text{ and } \phi_\eta(\xi, +0) = -\phi_\eta(\xi, -0)$$

and the second integral in Eq. (20) vanishes. ϕ_η at the profile is given in any case by the boundary conditions. Rewriting Eq. (22) one then obtains

$$\begin{aligned}
 Q(x,y) = & 2\pi\phi(x,y) - 2 \int_0^L \psi(\xi - x, -y) \phi_\eta(\xi, +0) d\xi \\
 & + (1/2) \iint_\Omega \psi_\xi(\xi - x, \eta - y) \phi_\xi^2(\xi, \eta) d\xi d\eta.
 \end{aligned} \tag{24}$$

The same equation is obtained for a symmetric flow if one considers only the upper half plane and uses instead of ψ the Green's function (whose η derivative vanishes at the axis)

$$\frac{1}{2}(\psi(\xi-x, \eta-y) - \psi(\xi-x, \eta+y)).$$

Eq. (24) is particularly simple because the unknown function appears only in the double integral.

In the usual integral equation method the direct use of the velocity components $u = \phi_x$ and $v = \phi_y$ is preferred. This is convenient because u and v are of direct physical interest. These equations are obtained by differentiating Eq. (22) with respect to x and y . Details are shown in Appendix III Eqs. (A18) and (A19). One obtains:

$$\begin{aligned} Q_x(x, y) = & 2\pi u(x, y) + \int_0^L (v(\xi, +0) - v(\xi, -0)) \psi_\xi(\xi-x, -y) d\xi \\ & + \int_0^L (u(\xi, +0) - u(\xi, -0)) \psi_\eta(\xi-x, -y) d\xi \\ & - (\pi/2) u^2(x, y) - 1/2 \iint_{\Omega} u^2(\xi, \eta) \psi_{\xi\xi}(\xi-x, \eta-y) d\xi d\eta \end{aligned} \quad (25)$$

$$\begin{aligned} Q_y(x, y) = & 2\pi v(x, y) + \int_0^L (v(\xi, +0) - v(\xi, -0)) \psi_\eta(\xi-x, -y) d\xi \\ & - \int_0^L (u(\xi, +0) - u(\xi, -0)) \psi_\xi(\xi-x, -y) d\xi \\ & - (1/2) \iint_{\Omega} u^2(\xi, \eta) \psi_{\xi\eta}(\xi-x, \eta-y) d\xi d\eta. \end{aligned} \quad (26)$$

Here, as before, Ω refers to the ξ, η plane punctured at the point $(\xi, \eta) = (x, y)$. The first integral on the right in each of these expressions is known because v is given as boundary condition. The second integrals do not extend over the wake. The last terms arise by differentiation of the double integral with respect to x or y . If one omits these terms then one obtains the integral equations of the linearized problem; and, for the symmetric flow, even its solution. The terms $\psi_{\xi\xi}$ and $\psi_{\xi\eta}$ which occur in the double integrals are singular at $(\xi, \eta) = (x, y)$. One cannot tell in advance how significant the double integral is in comparison to the

terms $(\pi/2)u^2(x,y)$. Nixon has found by numerical experimentation that a useful starting approximation for an iterative solution is obtained by omitting the double integral.

In a method of weighted residuals based on Eq. (21), one postulates that the expression $Q(x,y)$ vanishes for all weight functions (this means for all values of x and y). In a method of weighted residuals based on Eq. (25), one postulates that Q_x vanish for all values of x and y . At the end of Appendix II it is shown that Q is continuous at the shock. Equivalence between the formulations (Eqs. (21) and (25)) is therefore obtained if $Q = 0$ along some initial line which intersects all lines $y = \text{constant}$, for instance a line at $x = -\infty$. In this argument Eq. (26) does not play a role. In practical computations Eq. (26) may well be useful for the determination of the v field, but in theory the irrotationality condition $v_x = u_y$ serves equally well.

Eq. (25) can be derived from a number of different (although related) postulates. The flexibility in viewing the problems so obtained may sometimes be useful. In Appendix IV it is shown that Eq. (25) constitutes the weighted residual expression (not only the x derivative of such an expression) if one applies the weight function $-\psi_\xi$ to the potential equation.

Appendix V uses the formulation of the problem in terms of the velocity components u and v . It is shown that the expression (25) is obtained by applying the weight function $-\psi_\xi$ to the equation of conservation of mass, Eq. (12).

Appendix VI shows that Eq. (25) is obtained by applying the weight function ψ to the formulation of the problem in terms of u , Eq. (15). These results are not unexpected. However, the manipulations by which they are substantiated are not entirely straightforward.

Eq. (25) gives the integral equation formulation solely in terms of u and v . There is, however, a conceptual difficulty. In deriving this equation one has assumed that u and v are partial

derivatives of a potential. The existence of a potential is guaranteed if one constructs the v field pertaining to a given u field in the manner shown in Section III, for this construction is based on the irrotationality condition $u_y = v_x$. The continuity of the potential at the shock is expressed by the jump condition Eq. (14). The other condition to be observed at a shock is conservation of mass, expressed by Eq. (13). One will ask whether the integral equation formulation derived from Eq. (25) encompasses these two conditions. One might be inclined to dismiss this question by the observation that a serious error in the formulation of the problem as the omission of a shock condition would have been discovered during many practical applications made with the method. The theoretical answer is given at the end of Appendix V. The two shock conditions are indeed satisfied, provided that the expression Eq. (25) vanishes for all values of x and y .

Appendix VII gives even more specific information. It shows that the shock condition, Eq. (13), which expresses conservation of mass is already satisfied if Q_x vanishes for points (x,y) immediately upstream and downstream of the shock. For points (x,y) for which Eq. (26) is satisfied immediately upstream and downstream of the shock, one finds that a certain linear combination of the shock condition, Eqs. (13) and (14) is satisfied.

These results are not restricted to normal shocks.

There is no need to introduce separately shock conditions if one uses Eq. (25). To the contrary, if one would express the shock condition a second time by an independent postulate one might introduce ill conditioning into the formulation of the problem, because in the discretization which is always needed in practical work, one may inadvertently express the same conditions in two forms which are not completely identical.*

*The procedure proposed by Nixon in Ref. 6 might lead to this difficulty. The condition Eq. (29c) in this reference is a consequence of conditions Eq. (29a) and (29b) and can therefore not be used. In a later publication, Ref. (8), an equation corresponding to Eq. (29c) does not appear.

SECTION V

CONDITIONS AT THE SONIC LINE

In nearly all approaches to transonic flow (finite differences, finite elements, integral equations methods) one uses, implicitly or explicitly, the concept of weak equality. It is possible that expressions which solve a partial differential equation in the weak sense, fail to satisfy it pointwise. In the transonic case, solutions containing an expansion shock in addition to a compression shock thus become possible. One must introduce into the numerical approach the requirement that expansion shocks are not admitted. If one has an approximation to the flow field with a smooth transition from subsonic to supersonic flow and one studies perturbations to such a solution, then one finds that there are particular solutions which contain at the transition point, a logarithmic singularity. Only in the weak sense can expressions with such a singularity be regarded as solutions. Physically such perturbations are not admissible. One must impose the condition that at the sonic line the partial differential equation is satisfied in the strong sense. In the two-dimensional case this is simply expressed by the requirement that

$$Q_{yy} = 0. \quad (27)$$

It is shown in Ref.11 that the sonic operator used in the Murman Cole procedure can be interpreted in this manner.

In the framework of the integral equation formulation, the condition of a smooth transition through the sonic line is expressed in very ingenious but a rather indirect manner which has been introduced by Spreiter (Ref.2). In all integral equation methods one postulates, of course, that the weighted residual expression (25) be zero. This expression is written in the form

$$u - u^2/2 = R$$

or

$$u^2 - 2u = -2R. \quad (28)$$

This is a quadratic equation in which the right hand side contains the unknown function u . In the iteration procedure used to solve this problem the right hand side is evaluated from the preceding iteration step. Solving this quadratic equation, one obtains

$$u = 1 \pm \sqrt{1-2R}. \quad (29)$$

We mentioned before that at the sonic line $u = 1$. Therefore, one must have

$$R = 1/2.$$

Besides, in order for u to be real upstream and downstream of the sonic line, R must have a maximum at the sonic point. One therefore has

$$\partial R / \partial x = 0.$$

(In the subsonic region one then uses the lower, in the supersonic region the upper sign in Eq. (29)).

If the maximum of R is smaller than $1/2$, then U will never reach the value 1, and the transition from a subsonic to a supersonic flow can occur only in the form of an expansion shock. This formulation expresses indeed the requirement that u reaches the value 1 as one approaches the supersonic region from the subsonic side upstream.

In an iteration process there is always some arbitrariness regarding the terms relegated to the right hand side. The form of Eq. (25) might suggest that one uses (instead of Eq.(28)), an equation of the form

$$u - u^2/4 = R.$$

Then

$$u = 2 + \sqrt{4-4R}.$$

Here one finds that R must have a maximum at 1 in order for u to pass smoothly through the value 2 along a line $y = \text{constant}$. Of course, such a postulate is without interest, but one sees that only for a special choice of the iteration procedure will Eq. (28) yield conditions for the sonic line.

Spreiter and Oswatitsch, and Zierep and Nixon have devised workable procedures to incorporate this condition into the integral equation method. Yet, it has a somewhat artificial character; a combination of the integral equation method with finite difference or elements methods which becomes possible, if one regards the finite element method as one of weighted residuals, can lead to a more direct expression of the conditions at the sonic line, based on Eq. (27).

Nixon in Ref. 8 has given a formulation similar to Eq. (28) for the vicinity of the shock. Dividing Eq. (16) by $[u]_+^+$, one finds

$$\frac{u^+ + u^-}{2} \cos^2 \beta = 1$$

where we write temporarily that u^+ and u^- for the values of u , respectively, upstream and downstream of the shock. For the normal shock ($\beta = 0$) this amounts to an approximation to the Prandtl relation

$$w^+ w^- = a^{*2}$$

where w^+ and w^- denote the velocity upstream and downstream of the shock and a^* is the sonic velocity. It follows that in a formulation of Eq. (25) in the form

$$u - ((\cos^2 \beta)/2) u^2 = R.$$

R is continuous through the shock and the values of u obtained in each iteration step by solving this quadratic equation are connected by the proper shock relations.

SECTION VI

THE INTEGRAL EQUATION METHODS AND SOME EXTENSIONS

So far we have discussed expressions for weighted residuals formed with the weight functions ψ . If one has an expression for ϕ (or for u and v) for which the original differential equation and the shock conditions are satisfied, then the weighted residuals Q (or Q_x) will vanish for all values of (x,y) . One will remember that (x,y) are the parameters which characterize the individual weight functions. In practice the condition $Q_x = 0$ (Eq. (25)) will be imposed only at selected lines or selected points of the flow field. One can hope that the desired technical information (the pressure distribution over the profile) can be obtained by the use of a relatively small number of values (x,y) , because the weight functions used here (approximate fundamental solutions) have the property of making the expressions Q_x fairly insensitive against errors in the residual.

In the original form of the integral equation method the condition $Q_x = 0$ was imposed only for points of the profile, or perhaps points of the profile and some points of the x axis upstream and downstream. The expression Q_x Eq. (25) contains, besides local terms and expressions which can be computed from the boundary conditions, a double integral which contains values of u from the flow field. This is the only term in which a failure to satisfy the differential equation or the shock conditions appears. The values of u which occur in this double integral are now approximated in terms of the values of u at the x axis. Here one uses, besides the values of u , the values of u_y given at the profile by the boundary conditions, and, if one wants to have a better approximation, the fact that at the x axis (as everywhere else) the differential equation for u must be satisfied. The expressions which one chooses must vanish sufficiently quickly as one goes to infinity. From a practical point of view one prefers to choose formulae for u for which one of the integrations needed in the double integral can be carried out analytically.

Ultimately the double integral is reduced to a single integral which contains as unknowns, only the values of u at the points of the x axis under consideration. Substituting these expressions into Eq. (25) and for unsymmetric problems also in Eq. (26) one obtains one or two integral equations, which are subsequently solved by iteration. Under favorable circumstances the contribution of the double integral is small and one obtains in this manner results for the pressure distribution over the profile which are of sufficient accuracy for technical purposes.

A very attractive feature of this formulation lies in the fact that the possibility of the occurrence of shocks (in this approximation normal shocks) is put into evidence, if one solves Eq. (25) formally as a quadratic equation.

In cases with an embedded supersonic region u is definitely not small. At the sonic line one has for instance $u = 1$. A more accurate determination of u is therefore desirable. One can, for instance introduce for $u(x,y)$ within the field expression which satisfy at $x = 0$ the partial differential equation for u and also the first few of the derivatives of the differential equation with respect to y . This procedure is limited by the complexity of the expressions for u obtained in this manner.

To achieve a more accurate determination of u , Nixon and Hancock impose the condition $Q_x = 0$ not only along the x axis but also along some lines $y = \text{constant}$ within the field, and thus obtain a considerable improvement of the method.

According to the interpretation given in the present report, one then uses weight functions ψ pertaining to inner points (x,y) of the flow field. The author believes that other weight functions may serve equally well, if not better. To obtain good results for the pressure distribution at the profile by the extended integral equation method it is, of course, necessary that the residuals at points close to the profile be small. The weight functions ψ take into account residuals throughout the flow field,

the local residual at a point close to the profile may fail to vanish although $Q_x = 0$ at the point (x,y) under consideration. For points in the vicinity of the profile where u is not small, the use of a localized weight function therefore appears to be preferable to the conditions obtained from the integral equation formulation. Following this idea one uses finite difference or finite element formulations for a region which extends somewhat beyond the supersonic region.

Such a procedure has one additional advantage. The expressions Q_x contain contributions from all points of the flow field at which u is to be determined. In contrast, a finite difference formulation connects the value of u at a certain point (x,y) only with values at neighboring points. For numerical purposes this is a significant simplification.

One thus arrives at a hybrid procedure. For those parts of the flow field where u is small, one may take advantage of the fact, that the use of weight functions yields equations in which a dominant portion is directly expressed by linear theory, and the remaining nonlocal (and nonlinear) terms are small. For these expressions it may suffice if one describes the values of u which appear in these correction terms by means of a very small number of points (x,y) within the flow field. In the portion of the flow field where u is not small, one uses a finite difference or a finite element procedure. In this region one may even replace the simplified potential equation (Eq. 1) by a full potential equation.

Accordingly, we divide the flow field into an inner region in which we use a finite difference and a finite element procedure, and an outer region in which one uses the integral equation formulation (Figure 10). The inner region extends beyond the embedded supersonic region, but still it may be fairly small. The boundary of the region in which the integral equation formulation is applied is no longer confined to the x axis along this boundary, the values of v as well as the values of u are unknown. The derivations shown in the Appendix include cases of this kind. One obtains the following expression, Eq. (A17):

$$\begin{aligned}
Q_x(x,y) &= 2n(u(x,y) - (1/4)u^2(x,y) + \int_{\partial\Omega} v(\psi_\xi d\xi + \psi_\eta d\eta) \\
&\quad - \int_{\partial\Omega} (u - (1/2)u^2)\psi_\xi d\eta - u\psi_\eta d\eta \\
&\quad - (1/2) \iint_{\partial\Omega} \psi_\xi \psi_\eta u^2 d\xi d\eta = 0
\end{aligned} \tag{30}$$

Here the region Ω is the part of the flow field in which the integral equation formulation is applied, punctured at the point (x,y) . If (x,y) is a point of the contour of the region in which this formulation is applied, then one must carry out a limiting process particularly in the second integral over $\partial\Omega$. The result is found in Eq.(A48). It is probably practical not to consider the boundary $\partial\Omega$ in Figure 11 as a line which separates different regions of the flow field and for which a matching of the solutions must be carried out, but rather as a line which separates regions in which the residuals are weighted in a different fashion. For points of this boundary one may apply either the weighting of the residuals by means of the integral equation method (in this case one would apply Eq. (A48) or one may use the weighting applied in finite difference methods. Practical experience must show which procedure is more practical. One has the choice of including in the integral equation formulation the residuals that are due to the inner region (they are presumably small because one postulates that their local averages be zero); then the boundary between the two regions does not appear in the formulae, and the region Ω covers the entire flow field. Alternatively one can define Ω to be only the outer region. The latter form is probably preferable.

Because of the nonlinearity of the problem iterations cannot be avoided. But even in extensions of the method where one deals with linear equations one will probably use iterative procedures in the outer region because the number of unknowns is fairly large and the matrices which appear are not sparse. It can be expected that such iterations converge well. In the inner region one will also use iterations (for instance the Murman Cole iteration) at least in the initial stages of the procedure. After an initial

approximation has been attained one might think of a Newton procedure, in which the individual linear equations are solved by direct elimination. So far these important practical aspects have not been explored.

SECTION VII

INTEGRAL EQUATION FORMULATION AND FAR FIELD CONDITIONS

In practice the field for which a numerical procedure is actually carried out is finite. The effect of the field outside of its boundaries can be approximately taken into account by means of the integral equation formulation. For this purpose it is, of course, necessary to express the values of u in the distant field (that is outside of the boundaries of the computed field) in terms of the value of ϕ and ϕ_n or alternatively of u and v , at this boundary. The situation is analogous to that encountered in the original integral equation method, except that in the distant field the values of u are smaller and therefore the approximation is less critical. Of course, in this case the values of ϕ_n are not known in advance. If one chooses to use finite difference or finite element methods throughout the computed part of the flow field, then the integral equation formulation serves solely to formulate far field conditions.

In deriving the usual far field conditions one assumes that in the distant field the linearized potential equation is satisfied. In the formulation arising from the integral equation formulation also nonlinear terms in the flow differential equation for the distant field are taken into account. This is an improvement which ought to make it possible to reduce the size of the portion of the field in which the actual computations are carried out.

Source terms appear also in Klunker's formulation of the far field conditions (Ref. 12), but these sources lie within the computed parts of the flow field, while in the distant field it is assumed that the linearized potential equation is sufficiently accurate.

A first formulation of the far field condition is obtained from Eq. (A6). Here Ω is the distant field and $\partial\Omega_2$ its inner contour. This equation is evaluated for $\alpha = 1$. We quote:

$$\begin{aligned}
& \alpha \pi \phi(x, y) + \int_{\partial \Omega_2} \psi(\xi-x, \eta-y) [\phi_{\xi}(\xi, \eta) d\eta - \phi_{\eta}(\xi, \eta) d\xi] \\
& - \int_{\partial \Omega_2} \phi(\xi, \eta) (\psi_{\xi}(\xi-x, \eta-y) d\eta - \psi_{\eta}(\xi-x, \eta-y) d\xi) \\
& - \Gamma(\pi - \theta(x-L_1, y)) \\
& - (1/2) \iint_{\Omega} \psi(\xi-x, \eta-y) \frac{\partial}{\partial \xi} (\phi_{\xi}^2(\xi, \eta)) d\xi d\eta = 0
\end{aligned} \tag{31}$$

Provided that one has an approximation for the double integral one obtains a global relation between ϕ and its normal derivative along $\partial \Omega_2$. The expression contains a term due to the circulation.

A form which comes closer to the far field conditions derived (for more general problems) in Ref. 13, is obtained if one chooses the point (x, y) inside of the boundary $\partial \Omega_2$. One then must set $\alpha = 0$ in the last equation. The expression so obtained agrees with Eq. (39) of Ref. 13 except for the contributions of the nonlinear terms in the distant field Ω , and of the circulation. The latter term is missing because the author considers in this reference the potential after the contribution of the circulation has been subtracted.

To obtain, in the notation of the present article, an expression where ϕ denotes the potential with the circulation included, one writes Eq. (39) of Ref. 13 in the form

$$\begin{aligned}
& \int_{\partial \Omega_2} \{ \phi(\xi, \eta) - \Gamma[(1/2) - (\theta(\xi, \eta)/2\pi)] \} \{ (-\psi_{\xi}(\xi-x, \eta-y) d\eta \\
& + \psi_{\eta}(\xi-x, \eta-y) d\xi) - \int_{\partial \Omega_2} \psi(\xi-x, \eta-y) \{ -\frac{\partial}{\partial \xi} [\phi(\xi, \eta) - \Gamma[(1/2) \\
& - \theta(\xi, \eta)/2\pi]] d\eta + \frac{\partial}{\partial \eta} [\phi(\xi, \eta) - \Gamma[(1/2) - \theta(\xi, \eta)/2\pi]] d\xi \} = 0
\end{aligned}$$

Hence,

$$\begin{aligned}
& \int_{\partial\Omega_2} \phi(\xi, \eta) [-\psi_\xi(\xi-x, \eta-y) d\eta + \psi_\eta(\xi-x, \eta-y) d\xi] \\
& - \int_{\partial\Omega_2} \psi(\xi-x, \eta-y) [-\phi_\xi(\xi, \eta) d\eta + \phi_\eta(\xi, \eta) d\xi] \\
& - \Gamma \left\{ \int_{\partial\Omega_2} \left[(1/2) - (\theta(\xi, \eta)/2\pi) \right] \psi_\xi(\xi-x, \eta-y) d\eta + \psi_\eta(\xi-x, \eta-y) d\xi \right\} \\
& - \int_{\partial\Omega_2} \psi(\xi-x, \eta-y) \left[-\frac{\partial}{\partial \xi} \left((1/2) - (\theta(\xi, \eta)/2\pi) \right) d\xi \right. \\
& \left. + \frac{\partial}{\partial \eta} \left((1/2) - (\theta(\xi, \eta)/2\pi) \right) d\eta \right] \} = 0
\end{aligned} \tag{32}$$

Now one observes that $\theta(\xi, \eta)$ as well as $\psi(\xi-x, \eta-y)$ satisfies Laplace's equation in the region Ω . (One will remember that the singular point (x, y) lies outside of the region Ω .) The two integrals with the factor Γ extended over the whole contour $\partial\Omega$ therefore give zero. This contour consists of the large circle $\partial\Omega_1$, the contour separating the distant field from the computed part of the flow field $\partial\Omega_2$, and the two borders of a cut along the x axis $\partial\Omega_3$. It is shown in Appendix II that the integral $\partial\Omega_1$ vanishes. One therefore has

$$\int_{\partial\Omega_2} = - \int_{\partial\Omega_3} .$$

Most terms in the integrands vanish, since $\partial\Omega_3$ extends along the x axis; one obtains

$$\begin{aligned}
+ \Gamma \int_{\partial\Omega_3} \left[(1/2) - (\theta(\xi, \eta)/2\pi) \right] \psi_\eta(\xi-x, \eta-y) d\xi &= \Gamma \int_{L_1}^{\infty} \psi_\eta(\xi-x, \eta-y) d\xi \\
&= -\Gamma (\pi - \theta(x-L_1, y))
\end{aligned}$$

Hence, from Eq. (32)

$$\begin{aligned}
& \int_{\partial\Omega_2} \phi(\xi, \eta) [-\psi_{\xi}(\xi-x, \eta-y) d\eta + \psi_{\eta}(\xi-x, \eta-y) d\xi] \\
& - \int_{\partial\Omega_2} \psi(\xi-x, \eta-y) [-\phi_{\xi}(\xi, \eta) d\eta + \phi_{\eta}(\xi-x, \eta-y) d\xi] \\
& - \Gamma(\pi - \theta(x - L_1, y)) = 0.
\end{aligned} \tag{33}$$

This is Eq. (A6) for $\alpha = 0$ but without the source terms.

Source terms in the Laplace equation can be included also in the procedure of Ref. 13. One may do this by inspection, or by the following rationale. One divides the distant field into a very distant field in which no source terms are presented and in a moderately distant field in which sources play a role (Fig. 11). The inner boundary of the very distant field consists of a portion denoted by $\partial\Omega_6$ and a cut along the x axis extending from L_2 to infinity. Eq. (33) holds along $\partial\Omega_6$ (rather than $\partial\Omega_2$) which is, of course, the outer boundary of the moderately distant field. Now one applies Green's theorem to Eq. (5) with the weight function $\psi(\xi-x, \eta-y)$; (x, y) outside the very distant and moderately distant fields. Using Eq. (33), one arrives at an expression in which only the contribution of $\partial\Omega_2$ occurs. This is Eq. (31) with $\alpha = 0$.

If one solves for the inner field iteratively, by means of a Murman Cole iteration, say, then one needs local rather than global boundary conditions along $\partial\Omega_2$. To obtain such conditions we follow the argument used in Ref. 13.

Assume that we possess in the distant field a solution of

$$\phi_{xx} + \phi_{yy} - q(x, y) = 0$$

where as always

$$q = (1/2) \frac{\partial}{\partial x} (\phi_x^2).$$

We define an extension of this distant field throughout the inner field which satisfies $\phi_{xx} + \phi_{yy} = 0$. (Because of the

presence of the circulation one needs a cut along the x axis also in the inner field.) The values of ϕ match at the boundary between the inner and the outer field. There will be a jump of the normal derivative at the boundary between the two fields. This jump can be represented by a source distribution with the unknown intensity $f(s)$ (or $f(\sigma)$) where s (or σ) measures the distance along the boundary line. One thus obtains

$$\begin{aligned} \phi(x,y) = & (2\pi)^{-1} \left\{ \int_{\partial\Omega} f(\sigma) \psi(\xi(\sigma)-x, \eta(\sigma)-y) d\sigma \right. \\ & \left. + \int_{\Omega} q(\xi,\eta) \psi(\xi-x, \eta-y) d\xi d\eta \right\} + \Gamma((1/2) - \theta(x(s), y(s))/2\pi) \end{aligned} \quad (34)$$

Here Ω denotes the distant field and $\partial\Omega_2$ its inner boundary, the cut along the positive x axis is not part of $\partial\Omega_2$. The form in which the circulation enters is taken from Appendix II. It does not matter whether one writes $\theta(x,y)$ or $\theta(x-L,y)$ as long as the point where θ is singular, namely $x = L, y=0$, lies outside of the distant field (inside the inner field), for in the distant field, the difference $\theta(x-L,y) - \theta(x,y)$ can be represented by

$$\int_{\partial\Omega} f(\sigma) \psi(\xi(\sigma)-x, \eta(\sigma)-y) d\sigma.$$

A change of L therefore amounts to a change of the function $f(s)$.

The last expression and its normal derivative are evaluated for points $x(s)$ and $y(s)$ on $\partial\Omega$, the boundary between the distant field and the inner field for which the differential equation is integrated numerically. The function $f(s)$ must be determined in such a manner that the value of ϕ and ϕ_n from the outer field and from the computed part of the flow field match. The last equation can be used directly to determine ϕ . The evaluation of ϕ_n is shown in Appendix VIII. We repeat the definitions and the result. Let $\beta(s)$ be the angle of the contour $\partial\Omega$ of the distant field with the x axis. Then one obtains, according to Eq. (A52) and (A53),

$$\begin{aligned}
\phi_n(x(s), y(s)) &= (f(s)/2) \\
&+ (2\pi)^{-1} \left\{ \int_{\partial\Omega} -f(\sigma) \psi_n(\sigma, x(s), y(s)) \cos(\beta(\sigma) - \beta(s)) d\sigma \right. \\
&+ \int_{\partial\Omega} -f(\sigma) \psi_t(\sigma, x(s), y(s)) \sin(\beta(\sigma) - \beta(s)) d\sigma \\
&- \cos\beta(s) \iint_{\Omega} q(\xi, \eta) \psi_\xi(\xi - x(s), \eta - y(s)) d\xi d\eta \\
&+ \sin\beta(s) \iint_{\Omega} q(\xi, \eta) \psi_\eta(\xi - x(s), \eta - y(s)) d\xi d\eta \\
&\left. - (\Gamma/r(s)) \sin(\beta(s) - \theta(x(s), y(s))) \right\}
\end{aligned} \tag{35}$$

$$\begin{aligned}
\phi_t &= (2\pi)^{-1} \left\{ - \int_{\partial\Omega} f(\sigma) \psi_n(\sigma, x(s), y(s)) \sin(\beta(\sigma) - \beta(s)) d\sigma \right. \\
&+ \int_{\partial\Omega} f(\sigma) \psi_t(\sigma, x(s), y(s)) \cos(\beta(\sigma) - \beta(s)) d\sigma \\
&- \sin\beta(s) \iint_{\Omega} q(\xi, \eta) \psi_\eta(\xi - x, \eta - y) d\xi d\eta \\
&+ \cos\beta(s) \iint_{\Omega} q(\xi, \eta) \psi_\xi(\xi - x, \eta - y) d\xi d\eta \\
&\left. - (\Gamma/r) \cos(\beta(s) - \theta(x(s), y(s))) \right\}
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
r(s) &= (x(s)^2 + y(s)^2)^{1/2} \\
\theta(x, y) &= \arctg(y/x), \quad 0 \leq \theta \leq 2\pi \\
\psi_n(\sigma, x, y) &= \psi_\xi(\xi(\sigma) - x, \eta(\sigma) - y) \cos\beta(\sigma) + \psi_\eta(\xi(\sigma) - x, \eta(\sigma) - y) \sin\beta(\sigma) \\
\psi_t(\sigma, x, y) &= \psi_\xi(\xi(\sigma) - x, \eta(\sigma) - y) \sin\beta(\sigma) - \psi_\eta(\xi(\sigma) - x, \eta(\sigma) - y) \cos\beta(\sigma)
\end{aligned} \tag{37}$$

For special choices of the contour ($\beta = 0$, or $\beta = \pi/2$) this expression will simplify greatly. Some remarks regarding the approximation of q (in essence of u^2) which occur in the double integrals of Eqs. (31), (34), (35), and (36) will be made in the next section.

These formulae will be used in the following manner. Assume that one has an approximation to the function $f(s)$. One

then computes the values of ϕ and ϕ_n from Eq. (34) and (35). In ϕ_n one omits the contribution of the circulation. Using the value of ϕ as boundary condition at the outer edge of the computed flow field one computes the inner field, for instance by the Murman Cole iteration. It solves Eq. (5) with the boundary condition assigned at the profile and determines the circulation Γ so that the Kutta condition is satisfied. The expression for ϕ at the outer boundary must be modified if the circulation is modified. From the computation in the inner region one extracts the values of ϕ_n at its outer boundary, but without the contribution of the circulation. Let $\phi_{n,inner}$ and $\phi_{n,outer}$ be the value of ϕ_n (with circulation omitted) for the respective fields. A correction $\Delta f(s)$ to $f(s)$, which can be expected to lead to a convergent procedures, is then given by

$$\Delta f(s) = \phi_{n,outer}(s) - \phi_{n,inner}(s). \quad (38)$$

This discussion refers to computations in terms of the potential ϕ . The procedure for a procedure in terms of the velocity components u and v is very similar. The counterpart to Eq. (31) is Eq. (A48), if (x,y) is taken on the boundary between the inner and outer region, and Eq. (A43) with $\alpha = 0$ and $r=0$ if (x,y) is taken in the inner region. We quote:

$$\begin{aligned} & - \int_{\partial\Omega_2} \phi_{,n}(\xi-x, \eta-y) [u(\xi, \eta) - (1/2)u^2(\xi, \eta)] d\eta - v(\xi, \eta) d\xi] \\ & + \int_{\partial\Omega_2} \phi_{,n}(\xi-x, \eta-y) [u(\xi, \eta) d\xi + v(\xi, \eta) d\eta] \\ & - (1/2) \iint_{\Omega_2} \phi_{,n}(\xi-x, \eta-y) u^2(\xi, \eta) d\xi d\eta = 0 \end{aligned} \quad (39)$$

(x,y) in the computed region, and

$$\begin{aligned}
& + [u(x(s), y(s)) - (1/2) \sin^2 \beta(s) u^2(x(s), y(s))] \\
& - \int_{\partial \Omega_2} [u(\xi(\sigma), \eta(\sigma)) - (1/2) \cos^2 \beta(s) u^2(\xi(\sigma), \eta(\sigma))] \cdot n(\xi(\sigma) - x(s), \eta(\sigma) - y(s)) d\sigma + \\
& + \int_{\partial \Omega_2} [v(\xi, \eta) - (1/2) \sin \beta(s) \cos \beta(s) u^2(\xi(\sigma), \eta(s))] \quad (40) \\
& \phi_t(\xi(\sigma), x(\sigma)) d\sigma - (1/2) \iint_{\Omega} \nabla_{\xi \xi} (\xi - x, \eta - y) u^2(\xi, \eta) d\xi d\eta = 0
\end{aligned}$$

in the punctured outer region, (x, y) on the contour $\partial \Omega_2$. Notice that in these equations a contribution due to the circulation does not appear. (This holds only for the steady problem; in the unsteady problem u is not continuous across the wake.)

An alternative formulation, in which the distant field is represented by a source distribution in its interior and at its inner boundary is possible, also if one works with u and v . One computes for an approximate function $f(s)$ and an approximate circulation the values of ϕ_n and ϕ_t at the inner boundary of the distant field from Eqs. (35) and (36) and then expresses ϕ_t in terms of u and v . The inner field to which the finite difference procedure is applied is computed with the values of ϕ_t (expressed in u and v) as boundary condition. At the outer boundary again an adjustment of the circulation is needed. The subsequent procedure is the same as before.

SECTION VIII
APPROXIMATION OF u IN THE DISTANT FIELD

In application of the far field conditions an approximation for $q(x,y) = (1/2) \cdot (u^2)/x$ is needed for the distant field. One wants the computational effort for deriving such an approximation to be small. On the other hand the purpose of the far field conditions is to reduce the size of the portion of the flow field in which accurate computations must be carried out, and for this purpose a good approximation for q in the distant field is needed.

The problem of approximating q is of the same nature here as in the original formulations of the integral equation methods. But there one has some simplifying features. The boundary conditions for the profile are prescribed along the x axis. In the case of a symmetric flow field the solution for the linearized problem is directly expressed by quadratures. Unsymmetric flows are more complicated but at least the linearized flow field can be computed once and for all. In the present case the inner contour of the distant field is likely to be a rectangle. Along this boundary neither the normal derivative nor the potential is known. By the computation carried out in the inner field one obtains at the inner boundary of the distant field only a relation between the two quantities (or other related data, for instance, the velocity components u and v), and this relation changes from iteration to iteration. The relation so obtained must be combined with the far field conditions. Then one obtains ϕ and ϕ_n or u and v at the boundary. These are the quantities on which the estimation of q in the distant far field must be based.

At this stage the experience gained with the classical integral equation method may be helpful. To give an idea of what one might do, I describe here a procedure which has much in common with Nixon's extended integral equation method. For simplicity we assume that the inner contour of the distant field is given by two lines $y = \text{constant}$ at some distance above and below the profile (Fig.12). The computed part of the flow field

would then be a strip in the x, y plane. In practice one will, of course, cut off the strip by lines $x = \text{constant}$; then one needs additional conditions for these lines. We restrict ourselves to the simpler problem and derive expression for the upper part of the distant field.

Let y_1 be the value of y for the inner boundary of the distant field. Assume that u is known at $y = y_1$ and at another line $y = y_2$; $y_2 > y_1$. Along lines $x = \text{constant}$, u is then approximated by

$$u(x, y) = \frac{\Gamma}{2\pi} \frac{y}{(y^2 + x^2)} + \frac{a(x)}{y^2} + \frac{b(x)}{y^3}.$$

The first term is due to the circulation, the values of $a(x)$ and $b(x)$ must be determined in such a manner that u assumes the desired values at the lines $y = y_1$ and $y = y_2$. It is consistent with the character of the approximation if we simplify the last equation to

$$u(x, y) = \frac{\Gamma}{2\pi} y^{-1} + a(x)y^{-2} + b(x)y^{-3}$$

of course, with modified functions $a(x)$ and $b(x)$. One obtains

$$\begin{aligned} u(x, y) = & \Gamma/(2\pi y) \\ & + (y_1 - y_2)^{-1} \{ [(u(x, y_1) - (\Gamma/(2\pi y_1))) y_1^3 - (u(x, y_2) - (\Gamma/(2\pi y_2))) y_2^3] y^{-2} \\ & + [- (u(x, y_1) - (\Gamma/(2\pi y_1))) y_1^3 y_2 + (u(x, y_2) - (\Gamma/(2\pi y_2))) y_1 y_2^3] y^{-3} \} \end{aligned} \quad (41)$$

as can easily be verified. For other contours one might think of approximations of a similar character.

Assume now that we carry out a procedure based on Eq. (34). There one carries out an iteration for the computed part of the flow field in which the function $f(s)$ changes in each iteration step. We want to determine an approximation to $q(x, y)$ based on Eq. (41) which pertains to a fixed function $f(s)$. This is done

iteratively. Assume that one has some approximation to q . One then uses Eq. (34) to evaluate u for $y = y_1$ and y_2 . One has specifically:

$$u(x,y) = (2\pi)^{-1} \left[- \int_{\partial\Omega} f(\sigma) \phi_{\xi}(\xi(\sigma) - x, y_1 - y) d\sigma \right. \\ \left. - \iint_{\Omega} q(\xi, \eta) \phi_{\xi}(\xi - x, \eta - y) d\xi d\eta + Pr(x,y)^{-1} \sin\theta(x,y) \right] \quad (42)$$

These values of u are then substituted into Eq. (41), and with this equation values of q throughout the distant field are defined. With these values the computation is repeated. Of course, with a more complicated inner boundary of the distant field, the formulae for computing u at the pivotal lines of the distant field will become somewhat more complicated and one will in some regions use, instead of Eq. (41), somewhat different interpolation formulae.

It is likely that only very few iteration steps will be needed. The values of q so obtained are then used in the iteration process for the computed part of the flow field described in the preceding section. The key formula is Eq. (38).

The form of the far field conditions just discussed is probably best for steady flows because they yield local boundary conditions for ϕ (or ϕ_t) at the outer boundary of the computed flow field.

Because of their global character the far field conditions Eq. (31) can be used only for linear problems; for the number of simultaneous equations which one must solve is too large. Such linear problems arise if one tries to improve a sufficiently close approximation to the solution by a Newton-Raphson procedure, or if one computes small periodic disturbances to a steady flow field. In the latter case one may be forced to use these global conditions if the frequency is too high or the Mach number is too close to 1. (The question of how important such cases are for technical applications is left open.) We sketch the procedure applicable for these global boundary conditions.

One chooses, as solution strategy for the inner region, a procedure which yields, actually as an intermediate step, relations between the potential and its normal derivative at its outer boundary. These relations are combined with the far field conditions in their global form, Eq. (31). At this stage one must use some approximation for q , for instance $q = 0$. One then obtains the values of the potential and its normal derivative or rather the velocity components u and v at the outer boundary of the computed field (which is, of course, the inner boundary of the distant field). One then uses Eq. (A17) (with approximate values of u^2 in the field to compute u for $y = y_2$, Eq. (A17)).

$$\begin{aligned}
 & 2\pi u(x, y_2) + \int_{\partial\Omega_2} u(\xi, y_1) \psi_n(\xi - x, y_1 - y_2) d\xi \\
 & + \int_{\partial\Omega_2} v(\xi, y_1) [\psi_\xi(\xi - x, y_1 - y_2) d\xi - (\pi/2) u^2(x, y_2) \\
 & - (1/2) \iint_{\Omega} u^2(\xi, \eta) \psi_{\xi\xi}(\xi - x, \eta - y) d\xi d\eta.
 \end{aligned}$$

The values so obtained $u(x, y_2)$ together with the values of $u(x, y_1)$ with which this step starts are then substituted into Eq. (41) and the expression for u so obtained is used in the global form of the far field conditions, Eq. (31). Because of the changes in q which occur in this procedure, one will now obtain new values of u and v at the outer boundary of the computed field and with these values the approximation to the values of u in the distant field is improved.

We mentioned above that the procedure will be applied either to linear or linearized problems. Accordingly, there will be a linearization also in the far field conditions. It will depend upon the special circumstances whether a direct or an iterative approach is then preferable.

SECTION IX

CONCLUDING REMARKS

The report explores the integral equation method from a theoretical point of view in some depth, but without losing sight of its practical purpose. It is shown that the integral equation method can be regarded as a method of weighted residuals with rather unconventional weight functions.

The resulting equations contain terms obtained from linear theory which are solely determined by data at the boundary of the region under consideration and further nonlinear terms which depend upon flow field data. These latter terms are of smaller significance and can therefore be evaluated with reduced accuracy. In the original form of the integral equation method these field terms are approximated by means of boundary data. This method is particularly valuable for purely subsonic flows.

In the transonic region, nonlinear flow field data from the interior gain importance. To evaluate these data Nixon uses again the formulation of the integral equation method. Since the primary goal is to make the residuals in the partial differential equation small in the region close to the profile, it appears preferable to use these formulations which use local weights rather than the global weights of the integral equation method. Accordingly, the present paper advocates hybrid methods in which the integral equation formulation is used at points where the nonlinear terms are of small importance while finite difference or finite element methods are used in the region close to the profile. Such a procedure makes it possible to formulate the shock conditions and the conditions at the sonic line in a local form, which is closer to the nature of the problem at hand. The form in which these conditions are expressed by the integral equation method is very ingenious, it is true, but still somewhat awkward.

In an extreme development one uses the idea of the integral equation method only in the distant field in order to derive boundary conditions at the boundary between the distant

field and the computed part of the flow field. The far field conditions found in the literature assume that in the distant field the linearized potential equation holds. The method proposed here borrows from the integral equation method the idea, that the information about the distant field that is needed to take nonlinear terms into account, can be inferred from the data for potential and its normal derivative at the boundary between computed flow field and distant field. One can hope that such improved boundary conditions allow one to reduce the size of the computed part of the flow field without undue loss of accuracy.

The development has been carried out for two-dimensional problems. But extensions to the three-dimensional problem are not difficult. Such extensions have already been made by Nixon, but of course not for the hybrid form of the method and the far field conditions advocated in the present report.

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APPENDIX I
DERIVATION OF THE SHOCK CONDITIONS FROM THE
POTENTIAL EQUATION BY MEANS OF THE
CONCEPT OF WEAK EQUALITY

The starting point is the potential equation (5)

$$\frac{\partial}{\partial x} (\phi_x - (\phi_x^2/2)) + \frac{\partial}{\partial y} (\phi_y) = 0. \quad (A1)$$

Let $w(x,y)$ be a weight function of finite support (this means that w is different from zero in a finite region only). It is assumed that all derivatives which occur in the analysis exist and that w and its first derivative vanish at the boundary of the support region. The postulate that Eq. (A1) is satisfied in the weak sense is expressed by

$$\iint_{\Omega} w(x,y) \left[\frac{\partial}{\partial x} (\phi_x - (\phi_x^2/2)) + \frac{\partial}{\partial y} (\phi_y) \right] dx dy = 0$$

for every function $w(x,y)$. Carrying out the usual treatment of derivatives for generalized functions, one obtains by means of integrations by part

$$- \iint_{\Omega} [(\phi_x - (\phi_x^2/2)) \frac{\partial w}{\partial x} + \phi_y \frac{\partial w}{\partial y}] dx dy = 0.$$

The integral over the boundary of Ω vanishes because of the definition of w . The integrand is bounded even at the shock. The expression therefore remains unchanged if one excludes the shock by a strip of width 2ε and then lets ε tend to zero. Let the region so modified be denoted by Ω' (Fig. 1).

In the region Ω' the partial differential equation is satisfied pointwise. Now we carry out an integration by parts in the last equation. The boundary now includes portions $\partial\Omega_1$ and $\partial\Omega_2$ which extend upstream and downstream of the shock in the distance ε . Let

the shock be given by $x = x_s(y)$. Then one has for $\partial\Omega_1$

$$x = x_s(y) - \varepsilon$$

and for $\partial\Omega_2$

$$x = x_s(y) + \varepsilon.$$

The double integral which arises by the integration by parts vanishes because ϕ satisfies Eq. (A1) point wise in Ω' . The contribution of the outer contour of Ω' vanishes because of the definition of w . One therefore obtains

$$\int_{\partial\Omega_1 + \partial\Omega_2} w \{ [\phi_x - (\phi_x^2/2)] dy - \phi_y dx \} = 0$$

The direction of integration is to be chosen so that the region of integration lies to the left. This means that one proceeds along $\partial\Omega_1$ and $\partial\Omega_2$ in opposite directions. One has, because of the definition of w

$$\lim_{\varepsilon \rightarrow 0} w(x_s(y) - \varepsilon, y) = \lim_{\varepsilon \rightarrow 0} w(x_s(y) + \varepsilon, y) = w(x_s, y)$$

Using the notation of Section III one therefore obtains

$$\int_{\partial\Omega_1} w(x_s(y), y) \{ [\phi_x - (\phi_x^2/2)]_-^+ dy - [\phi_y]_-^+ dx \} = 0$$

In order for this equation to be correct for any choice of w , one must have

$$[\phi_x - (\phi_x^2/2)]_-^+ - [\phi_y]_-^+ \frac{dx_s}{dy} = 0.$$

APPENDIX II
THE EXPRESSION $Q(x,y)$

We have defined in Eq. (21)

$$Q(x,y) = \iint_{\Omega} \left[\frac{\partial}{\partial \xi} (\phi_{\xi} - (\phi_{\xi}^2/2)) + \frac{\partial}{\partial \eta} (\phi_{\eta}) \right] \psi(\xi-x, \eta-y) d\xi d\eta$$

At the moment Ω is some region of the ξ, η plane in which ψ satisfies the Laplace equation. Its boundary is denoted by $\partial\Omega$. The region may include shocks. One obtains in a familiar manner by carrying out integrations by part

$$\begin{aligned} Q(x,y) &= \int_{\partial\Omega} \psi(\xi-x, \eta-y) [\phi_{\xi}(\xi, \eta) d\eta - \phi_{\eta}(\xi, \eta) d\xi] \\ &\quad - \int_{\partial\Omega} \phi(\xi, \eta) [\psi_{\xi}(\xi-x, \eta-y) d\eta - \psi_{\eta}(\xi-x, \eta-y) d\xi] \\ &\quad - (1/2) \iint_{\Omega} \psi(\xi-x, \eta-y) \frac{\partial}{\partial \xi} (\phi_{\xi}^2(\xi, \eta)) d\xi d\eta \end{aligned} \quad (A2)$$

A further double integral with integrand $\phi(\psi_{\xi\xi} + \psi_{\eta\eta})$ vanishes because ψ satisfies the Laplace equation. The integration around the contour $\partial\Omega$ is performed in the direction for which the interior of Ω lies to the left. One has

$$\begin{aligned} &- (1/2) \iint_{\Omega} \psi(\xi-x, \eta-y) \frac{\partial}{\partial \xi} (\phi_{\xi}^2) d\xi d\eta = - \frac{1}{2} \int_{\partial\Omega} \psi(\xi-x, \eta-y) \phi_{\xi}^2 d\eta \\ &+ (1/2) \iint_{\Omega} \psi_{\xi}(\xi-x, \eta-y) \phi_{\xi}^2(\xi, \eta) d\xi d\eta \end{aligned} \quad (A3)$$

The form of the right hand side in Eq. (A3) may sometimes be preferable because it eliminates one differentiation in the unknown function ϕ_{ξ}^2 . It thus becomes applicable, even if, in some approximation, $\partial(\phi_{\xi}^2)/\partial\xi$ contains delta functions. The integrands of the double integrals are singular at the point $(\xi, \eta) = (x, y)$.

We choose for Ω a region of the ξ, η plane punctured at the point $(\xi, \eta) = (x, y)$ (Fig. 2). Its boundary is $\partial\Omega$. Part of $\partial\Omega$ at a great distance, to be denoted by $\partial\Omega_1$ is given by a circle with a radius R . Subsequently, the limiting process $R \rightarrow \infty$ will be made. In the familiar forms of the integral equation method the profile, extending from $\xi = 0$ to $\xi = L$, is replaced by a cut along the ξ -axis. To prepare for a modification, we allow for the possibility that the part of $\partial\Omega$ which is usually given by the two borders of this cut lies at some distance from the profile. This part of $\partial\Omega$ is denoted by $\partial\Omega_2$. The wake is excluded by a cut along the ξ -axis. The two borders of this cut form the boundary $\partial\Omega_3$. They are joined to $\partial\Omega_2$ either at the trailing edge $\xi = L$ or at some point further downstream $\xi = L_1$. The junction with $\partial\Omega_1$ lies at $\xi = R$. The small circle, by which the point $(\xi, \eta) = (x, y)$ is excluded is denoted by $\partial\Omega_4$.

In the presence of a circulation the term of the potential which are dominant at a large distance have the form

$$\phi = \text{const} - \Gamma\theta/2\pi$$

where Γ is the circulation and

$$\theta(\xi, \eta) = \arctg(\eta/\xi); \quad 0 \leq \theta \leq 2\pi.$$

The choice of the constant in the above equation has no effect on the flow field, because the flow differential equation contains only derivatives of ϕ . Accordingly, we can choose (for R large)

$$\phi = \Gamma((1/2) - (\theta/2\pi)). \quad (\text{A4})$$

As R tends to infinity, ϕ_ξ and ϕ_η tend to zero strongly enough that for the boundary $\partial\Omega_1$ the first integral on the right of Eq. (A2) vanishes. Regarding the second integral one observes that

$$\psi_\xi d\eta - \psi_\eta d\xi = (d\psi/dn) d\sigma$$

where d/dn denotes the derivative in the direction of the outer normal and $d\sigma$ is the arc length of $\partial\Omega$. For R sufficiently large

and (x, y) bounded, one has

$$\psi \sim \log R$$

$$\frac{d\psi}{dn} = R^{-1}$$

$$d\sigma = R d\theta.$$

One then obtains

$$- \int_{\partial\Omega_1} \phi(d\psi/dn) ds = \Gamma(\theta/2) - (\theta^2/4\pi) \Big|_0^{2\pi} = 0.$$

The contribution of the contour $\partial\Omega_1$ vanishes, if one chooses the constant in ϕ , which is arbitrary, according to Eq. (A4).

No simplifications arise in the integral for a general choice of the portion of the contour $\partial\Omega_2$. In cases where $\partial\Omega_2$ is represented by the two borders of a cut along the x axis (from $\xi=0$ to $\xi=L$) one can treat $\partial\Omega_2$ and $\partial\Omega_3$ together. On the lower side of the cut one moves in the negative, on the upper side in the positive ξ direction. In this case $d\eta=0$ and one obtains

$$\begin{aligned} & - \int_{\partial\Omega_2 + \partial\Omega_3} \psi(\xi-x, \eta-y) \phi_\eta(\xi, \eta) d\xi = \\ & - \int_0^L \psi(\xi-x, -y) \{ \phi_\eta(\xi, +0) - \phi_\eta(\xi, -0) \} d\xi \end{aligned}$$

Here we have written

$$\begin{array}{ll} \lim_{\eta \rightarrow 0} \phi_\eta(\xi, \eta) = \phi_\eta(\xi, +0), & \lim_{\eta \rightarrow 0} \phi_\eta(\xi, \eta) = \phi_\eta(\xi, -0) \\ \eta > 0 & \eta < 0 \end{array}$$

The wake does not appear in the integral because there ϕ_η is continuous. At the upper and lower side of the profile, ϕ_η is given by the boundary conditions. For a symmetric profile one has

$$\phi_\eta(\xi, -0) = -\phi_\eta(\xi, +0).$$

If $\partial\Omega_2 + \partial\Omega_3$ is given by the two borders of a cut along the ξ axis, then one obtains for the second integral in Eq. (A2)

$$\int_{\partial\Omega_2 + \partial\Omega_3} \phi(\xi, \eta) \psi_\eta(\xi - x, \eta - y) d\xi = \int_0^\infty [\phi(\xi, +0) - \phi(\xi, -0)] \psi_\eta(\xi - x, -y) d\xi$$

At the profile the difference $\phi(\xi, +0) - \phi(\xi, -0)$ is not known. For a symmetric flow field it vanishes along $\partial\Omega_2 + \partial\Omega_3$. In the wake ($\partial\Omega_3$) this difference is given by the circulation. Substituting, also the expression for ψ , Eq. (18), one obtains

$$+ \Gamma \int_{L_1}^\infty \psi_\eta(\xi - x, -y) d\xi = -\Gamma \theta(x - \xi, y) \Big|_{L_1}^\infty = \Gamma [\theta(x - L_1, y) - \pi]$$

where

$$\theta(x - \xi, y) = \arctg(y/(x - \xi)),$$

$$0 \leq \theta(x - \xi, y) < 2\pi.$$

The contour integral around the singular point is treated in a familiar way. Let

$$\xi - x = r \cos \theta$$

$$\eta - y = r \sin \theta$$

The first integral over $\partial\Omega_4$ in Eq. (A2) vanishes, because the singularity $\psi = \log r$ is too weak. For the evaluation of the second integral we note that

$$\psi_\xi = r^{-1} \cos \theta, \quad \psi_\eta = r^{-1} \sin \theta$$

$$d\xi = -r \sin \theta, \quad \psi_\eta = r \cos \theta$$

The integration proceeds in the negative θ direction. Therefore

$$\lim_{r \rightarrow 0} \left(- \int_{\partial\Omega_4} \phi(\xi, \eta) [\psi_\xi(\xi - x, \eta - y) d\eta - \psi_\eta(\xi - x, \eta - y) d\xi] \right) = 2\pi \phi(x, y)$$

One thus obtains

$$\begin{aligned}
 Q(x,y) = & 2\pi\phi(x,y) + \int_{\partial\Omega_2} \psi(\xi-x, \eta-y) [\phi_\xi d\eta - \phi_\eta d\xi] \\
 & - \int_{\partial\Omega_2} \phi(\xi, \eta) [\psi_\xi(\xi-x, \eta-y) d\eta - \psi_\eta(\xi-x, \eta-y) d\xi] \\
 & - \Gamma(\pi - \theta(x-L_1, y)) - (1/2) \iint_{\Omega} \psi(\xi-x, \eta-y) \frac{\partial}{\partial \xi} (\phi_\xi^2(\xi, \eta)) d\xi d\eta
 \end{aligned} \tag{A5}$$

Eq. (22) is obtained by specializing $\partial\Omega_2$ to a cut along the x axis.

A special limiting process familiar from potential theory is needed if the point x, y approaches a point x_c, y_c of the contour along the inner normal. One then treats a vicinity of the point x_c, y_c separately. Let $\partial\Omega'$ be a small portion of the contour $\partial\Omega$ which contains the point x_c, y_c , and let $\partial\Omega'' = \partial\Omega - \partial\Omega'$. For $(x,y) = (x_c, y_c)$ the integrand in

$$\int_{\partial\Omega''} \phi(\xi, \eta) [\psi_\xi(\xi-x_c, \eta-y_c) d\eta - \psi_\eta(\xi-x_c, \eta-y_c) d\xi]$$

remains bounded provided that $\partial\Omega$ has finite curvature at (x_c, y_c) . We therefore can immediately make the limiting process in which the excluded portion tends to zero. Discussions familiar from potential theory give

$$\lim_{(x,y) \rightarrow (x_c, y_c)} \int_{\partial\Omega'} \phi(\xi, \eta) [\psi_\xi(\xi-x, \eta-y) d\eta - \psi_\eta(\xi-x, \eta-y) d\xi] = -\phi(x_c, y_c)$$

Including also cases where the point (x,y) lies inside of $\partial\Omega_2$ we rewrite Eq. (A5) in the form

$$\begin{aligned}
 Q(x,y) = & \alpha\pi\phi(x,y) + \int_{\partial\Omega_2} \psi(\xi-x, \eta-y) [\phi_\xi(\xi, \eta) d\eta - \phi_\eta(\xi, \eta) d\xi] \\
 & - \int_{\partial\Omega_2} \phi(\xi, \eta) [\psi_\xi(\xi-x, \eta-y) d\eta - \psi_\eta(\xi-x, \eta-y) d\xi] \\
 & - \Gamma(\pi - \theta(x-L_1, y)) - (1/2) \iint_{\Omega} \psi(\xi-x, \eta-y) \frac{\partial}{\partial \xi} (\phi_\xi^2(\xi, \eta)) d\xi d\eta
 \end{aligned} \tag{A6}$$

where $\alpha = 2$ for (x,y) inside Ω
 $\alpha = 1$ for (x,y) on $\partial\Omega_2$
 $\alpha = 0$ for (x,y) outside Ω .

One has (in the second integral over $\partial\Omega_2$)

$$\lim_{\substack{x(s) \rightarrow \xi(\sigma) \\ y(s) \rightarrow \eta(\sigma)}} \left\{ \psi_\alpha(\xi(\sigma) - x(s), \eta(\sigma) - y(s)) \frac{d\eta(\sigma)}{d\sigma} - \psi_{\alpha+1}(\xi(\sigma) - x(s), \eta(\sigma) - y(s)) \frac{d\xi(\sigma)}{ds} \right\} = (1/2)R(\sigma)^{-1}$$

where $R(\sigma)$ is the radius of curvature at $\xi(\sigma), \eta(\sigma)$, positive if Ω is convex.

The expression $Q(x,y)$ is continuous along a line $y = \text{const}$ as the point (x,y) passes through a shock. This is seen in the following manner. We have postulated continuity of ϕ . In the integrals over $\partial\Omega_2$, ψ and its derivatives are continuous, because the shock is not part of $\partial\Omega_2$ and therefore $(x,y) \neq (\xi,\eta)$. A discontinuity can therefore be introduced only by the double integral. It would be caused by the jump of $\partial\Omega^2$ at the point $(\xi,\eta) = (x,y)$ if the point (x,y) passes through a shock. For a discussion it suffices if one considers the critical integral for the vicinity of some point of the shock $x = x_s(y)$ with $\phi_\xi = 1$ upstream and $\phi_\xi = 0$ downstream of the shock. We enclose this point by a rectangular contour (Fig. 3). Distance points of Ω will not give a discontinuity in Q , therefore it suffices if one evaluates the integral

$$(1/2) \iint_{\Omega} \psi_\xi(\xi - x, \eta - y) d\xi d\eta$$

once for $(x,y) = (x_s - \epsilon, y)$ and a second time for $(x,y) = (x_s + \epsilon, y)$ and examine the difference in the limit $\epsilon = 0$. The integration is to be carried out for the shaded area of Fig. 3. One obtains, by carrying out the integration with respect to ξ

$$(1/2) \left[- \int_{\eta_1}^{\eta_2} \psi(\xi_1 - x_s + \varepsilon, \eta - y) d\eta + \int_{\eta_1}^{\eta_2} \psi(\xi_s(\eta) - x_s + \varepsilon, \eta - y) d\eta \right]$$

in the second case with ε replaced by $-\varepsilon$. The first integral which comes from the left boundary of the rectangular box, will not cause a singularity as the sign of ε changes. In the second integral we place the origin of ξ' , η' system into the point (x_s, y) and approximate $\xi_s'(\eta)$ by $\varepsilon_s' = -\operatorname{tg} \beta \eta'$, where β is the angle of the normal to the shock at the point in question with the x axis. Finally we substitute ψ ; one then obtains

$$- (1/4) \int_{\eta_1}^{\eta_2} \log((- \eta' \operatorname{tg} \beta + \varepsilon)^2 + \eta'^2) d\eta'$$

This integral can be evaluated in closed form. The resulting expression is continuous as ε passes through zero.

APPENDIX III
DERIVATIVES OF $Q(x, y)$

After replacing the last term by means of Eq. (A3) one obtains from Eq. (A5) by differentiation with respect to x

$$\begin{aligned}
 Q_x(x, y) = & 2\pi\phi_x(x, y) + \int_{\partial\Omega_2} -\psi_\xi(\xi-x, \eta-y) [\phi_\xi d\eta - \phi_\eta d\xi] \\
 & - \int_{\partial\Omega_2} \phi(\xi, \eta) [-\psi_{\xi\xi}(\xi-x, \eta-y) d\eta + \psi_{\xi\eta}(\xi-x, \eta-y) d\xi] \\
 & - \Gamma \int_{L_1}^{\infty} \psi_{\xi\eta}(\xi-x, -y) d\xi \\
 & - (1/2) \int_{\partial\Omega_2} -\psi_\xi(\xi-x, \eta-y) \phi_\xi^2(\xi, \eta) d\eta \\
 & + (1/2) \frac{\partial}{\partial x} \iint_{\Omega} \psi_\xi(\xi-x, \eta-y) \phi_\xi^2(\xi, \eta) d\xi d\eta \quad (A7)
 \end{aligned}$$

In the expression with the factor Γ we have substituted the form in which it originally appeared in Appendix II. One can carry out the integration with respect to ξ and then obtains

$$-\Gamma \int_{L_1}^{\infty} \psi_{\xi\eta}(\xi-x, -y) d\xi = \Gamma \psi_\eta(L_1-x, -y) \quad (A8)$$

In the second integral over $\partial\Omega_2$ we use the relation $\psi_{\xi\xi} = -\psi_{\eta\eta}$. For the differentiation along a contour given by

$$\xi = \xi_c(\eta)$$

an operator

$$\frac{D_c}{d\eta} = \frac{\partial}{\partial \eta} + \frac{d\xi_c}{d\eta} \cdot \frac{\partial}{\partial \xi} \quad (A9)$$

is introduced. One then obtains for the second integral over $\partial\Omega_2$

$$\begin{aligned} & - \int_{\partial\Omega_2} \phi(\xi, \eta) [-\psi_{\xi\xi}(\xi-x, \eta-y) d\eta + \psi_{\xi\eta}(\xi-x, \eta-y) d\xi] \\ & = - \int_{\partial\Omega_2} \phi(\xi, \eta) \frac{D_c}{d\eta} (\psi_\eta) d\eta = - \phi(\xi, \eta) \psi_\eta \Big|_{\text{limits}} + \int_{\partial\Omega_2} \psi_\eta \frac{D_c}{d\eta} (\phi(\xi, \eta)) d\eta \\ & = -\phi(\xi, \eta) \psi_\eta \Big|_{\text{limits}} + \int_{\partial\Omega_2} \psi_\eta (\phi_\eta(\xi, \eta) d\eta + \phi_\xi(\xi, \eta) d\xi) \end{aligned}$$

The contour $\partial\Omega_2$ starts and ends at the point $\xi = L_1$, $\eta = 0$ where the two borders of the wake contour $\partial\Omega_3$ start or end. There the terms for the limits are canceled by the expression Eq. (A8).*

We write

$$\phi_x = u, \quad \phi_y = v \quad (A10)$$

The integrals over $\partial\Omega_2$ in Eq. (A7) then contract to

$$\begin{aligned} & \int_{\partial\Omega_2} -(u(\xi, \eta) - (1/2)u^2(\xi, \eta)) \psi_\xi(\xi-x, \eta-y) d\eta + u(\xi, \eta) \psi_\eta(\xi-x, \eta-y) d\xi \\ & + \int_{\partial\Omega_2} v(\xi, \eta) [\psi_\xi(\xi-x, \eta-y) d\xi + \psi_\eta(\xi-x, \eta-y) d\eta] \end{aligned} \quad (A11)$$

*This simplification occurs only for steady plane flows.

In evaluating the derivative of the double integral in Eq. (A7), some precautions are necessary because of the singularity of the integrand. In the original definition of the region Ω the singular point $(\xi, \eta) = (x, y)$ is excluded by a small circle of radius ϵ which moves as the point (x, y) moves. This must be taken into account in the differentiation with respect to x or y (up to now the exclusion of the point has not mattered in the evaluation of the double integral). We divide the region Ω into two regions by surrounding the point (x, y) by a box of finite dimensions (Fig. 4). The boundaries of the box remain fixed while the differentiation with respect to x or y is carried out. In the outside region Ω_1 the differentiation can immediately be carried out under the integral sign:

$$(1/2) \frac{\partial}{\partial x} \iint_{\Omega_1} \psi_{\xi}(\xi-x, \eta-y) u^2(\xi, \eta) d\xi d\eta = -(1/2) \iint_{\Omega_1} \psi_{\xi\xi}(\xi-x, \eta-y) u^2(\xi, \eta) d\xi d\eta \quad (A12)$$

For the evaluations within the box (region Ω_2) we set

$$\begin{aligned} \xi_1 &= \xi - x \\ \eta_1 &= \eta - y \\ r &= (\xi_1^2 + \eta_1^2)^{1/2} \end{aligned} \quad (A13)$$

The boundaries of the region $\partial\Omega_2$ are given by

$$\xi = x_1, \text{ and } \xi = x_2, \quad x_1 < x < x_2$$

$$\eta = y_1, \text{ and } \eta = y_2, \quad y_1 < y < y_2$$

and the small circle with radius ϵ around the point (x, y) . Let temporarily

$$I = (1/2) \frac{\partial}{\partial x} \int_{\xi=x_1}^{x_2} \int_{\eta=y_1}^{y_2} \psi_{\xi}(\xi-x, \eta-y) u^2(\xi, \eta) d\xi d\eta \quad r > \epsilon$$

then, after substitution of Eqs. (A13)

$$I = (1/2) \frac{\partial}{\partial x} \left[\lim_{\epsilon \rightarrow 0} \int_{\xi_1=x_1-x}^{x_2-x} \int_{\eta_1=y_1-y}^{y_2-y} \psi_{\xi}(\xi_1, \eta_1) u^2(\xi_1+x, \eta_1+y) d\xi_1 d\eta_1 \right]$$

In the $\xi_1 \eta_1$ system the small circle around the singular point is fixed, even if x or y changes. The function u^2 depends upon two arguments (originally ξ and η). For clarity, we denote differentiation with respect to its first or second argument, respectively, by subscripts 1 and 2. Now we carry out the differentiation with respect to x in the last expression

$$I = (1/2) \left[\int_{\eta_1=y_1-y}^{y_2-y} \psi_{\xi}(x_1-x, \eta_1) u^2(x_1, \eta_1+y) d\eta_1 - \int_{\eta_1=y_1-y}^{y_2-y} \psi_{\xi}(x_2-x, \eta_1) u^2(x_2, \eta_1+y) d\eta_1 \right. \\ \left. + \lim_{\epsilon \rightarrow 0} \int_{\xi_1=x_1-x}^{x_2-x} \int_{\eta_1=y_1-y}^{y_2-y} \psi_{\xi}(\xi_1, \eta_1) u^2(\xi_1+x, \eta_1+y) d\xi_1 d\eta_1 \right]$$

Returning to the original variables ξ and η one obtains

$$I = (1/2) \left[\int_{\eta=y_1}^{y_2} \psi_{\xi}(x_1-x, \eta-y) u^2(x_1, \eta) d\eta - \int_{\eta=y_1}^{y_2} \psi_{\xi}(x_2-x, \eta-y) u^2(x_2, \eta) d\eta \right. \\ \left. + \lim_{\epsilon \rightarrow 0} \int_{\xi=x_1}^{x_2} \int_{\eta=y_1}^{y_2} \psi_{\xi}(\xi-x, \eta-y) (\partial(u^2(\xi, \eta)/\partial \xi) d\xi d\eta \right] \quad (A14)$$

In the double integral an integration by parts with respect to ξ is carried out. Let $\partial\Omega'_2$ be the circle $(\xi-x)^2 + (\eta-y)^2 = \epsilon^2$.

One obtains

$$I = - \int_{\eta=y_1}^{y_2} \psi_{\xi}(x_1-x, \eta-y) u^2(x_1, \eta) d\eta + \int_{\eta=y_1}^{y_2} \psi_{\xi}(x_2-x, \eta-y) u^2(x_2, \eta) d\eta \quad (A15)$$

$$\lim_{\epsilon \rightarrow 0} \int_{\partial\Omega'_2} \psi_{\xi}(\xi-x, \eta-y) u^2(\xi, \eta) d\eta - \lim_{\epsilon \rightarrow 0} \iint_{\Omega'_2} \psi_{\xi\xi}(\xi-x, \eta-y) u^2(\xi, \eta) d\xi d\eta$$

The first two integrals in Eq. (A15) cancel the first two integrals in Eq. (A14). We introduce temporarily

$$\xi - x = r \cos \theta$$

$$\eta - y = r \sin \theta$$

Then along $\partial\Omega_2$

$$d\eta = \epsilon \cos \theta d\theta$$

Moreover

$$\psi_\xi = \epsilon^{-1} \cos \theta$$

The integration around $\partial\Omega_2$ proceeds in the negative θ direction. One obtains

$$\lim_{\epsilon \rightarrow 0} \int_{\partial\Omega_2} \psi_\xi (\xi - x, \eta - y) u^2(\xi, \eta) d\eta = -\pi u^2(x, y)$$

The double integrals in Eqs. (A15) and (A12) can be combined. One finally finds

$$\begin{aligned} (1/2) \frac{\partial}{\partial x} \left[\iint_{\Omega} \psi_\xi (\xi - x, \eta - y) u^2(\xi, \eta) d\xi d\eta \right] = \\ -\pi/2 u^2(x, y) - (1/2) \iint_{\Omega} \psi_{\xi\xi} (\xi - x, \eta - y) u^2(\xi, \eta) d\xi d\eta \end{aligned} \quad (A16)$$

Substituting these results into Eq. (A7) one obtains

$$Q_x(x, y) = 2\pi u(x, y) + \quad (A17)$$

$$\int_{\partial\Omega_2} [u(\xi, \eta) - (1/2)u^2(\xi, \eta)] \psi_\xi (\xi - x, \eta - y) d\eta + u(\xi, \eta) \psi_\eta (\xi - x, \eta - y) d\xi$$

$$+ \int_{\partial\Omega_2} v(\xi, \eta) [\psi_\xi (\xi - x, \eta - y) d\xi + \psi_\eta (\xi - x, \eta - y) d\eta]$$

$$- (\pi/2) u^2(x, y) - (1/2) \iint_{\Omega} u^2(\xi, \eta) \psi_{\xi\xi} (\xi - x, \eta - y) d\xi d\eta$$

Specializing this equation to the usual case, where Ω_2 is given by the two borders of the cut made along the profile (from $x = 0$ to $x = L$), one obtains:

$$\begin{aligned}
 Q_x(x, y) = & 2\pi u(x, y) + \int_0^L [(u(\xi, +0) - u(\xi, -0))] \psi_\eta(\xi - x, -y) d\xi \\
 & + \int_0^L [v(\xi, +0) - v(\xi, -0)] \psi_\xi(\xi - x, -y) d\xi - \\
 & - (\pi/2) u^2(x, y) - (1/2) \iint_{\Omega} u^2(\xi, \eta) \psi_{\xi\xi}(\xi - x, \eta - y) d\xi d\eta
 \end{aligned} \tag{A18}$$

A corresponding procedure can be carried out for Q_y . For the specialized region Ω one obtains

$$\begin{aligned}
 Q_y(x, y) = & 2\pi v(x, y) + \int_0^L [v(\xi, +0) - v(\xi, -0)] \psi_\eta(\xi - y, -y) d\xi \\
 & - \int_0^L [u(\xi, +0) - u(\xi, -0)] \psi_\xi(\xi - x, -y) d\xi \\
 & - (1/2) \iint_{\Omega} u^2(\xi, \eta) \psi_{\xi\eta}(\xi - x, \eta - y) d\xi d\eta
 \end{aligned} \tag{A19}$$

APPENDIX IV

DERIVATION OF THE EXPRESSION $Q_x(x,y)$ BY APPLYING THE WEIGHT FUNCTION $-\psi_\xi$ TO THE POTENTIAL EQUATION

The weight function $-\psi_\xi$ has a singularity at the point $(\xi, \eta) = (x, y)$. The question arises to what extent this fact imposes a restriction on the form of the residuals to which this weight function can be applied. Introducing, as before

$$\xi - x = r \cos \theta, \quad \eta - y = r \sin \theta$$

one has

$$-\psi_\xi = -r^{-1} \cos \theta, \quad -\psi_\eta = -r^{-1} \sin \theta$$

We postulate that the residuals to which this weight function is applied be continuous and that its first derivative exist at the point (x, y) . Such a residual can then be written as

$$R(\xi, \eta) = R_0 + R_\xi r \cos \theta + R_\eta r \sin \theta + \text{terms of higher order}$$

Here R_0 , R_ξ and R_η are constants. One then considers expressions

$$\begin{aligned} & - \iint_{r>\epsilon} (R_0 + R_\xi r \cos \theta + R_\eta r \sin \theta) \psi_\xi \, d\xi \, d\eta \\ &= - \iiint_{\substack{r>\epsilon \\ 0 \leq \theta \leq 2\pi}} (R_0 + R_\xi r \cos \theta + R_\eta r \sin \theta) r^{-1} \cos \theta \, r \, dr \, d\theta \\ &= - \int_{\theta=0}^{2\pi} [r R_0 \cos \theta + (r^2/2) (R_\xi \cos^2 \theta + R_\eta \sin \theta \cos \theta)] \bigg|_{r=\epsilon}^{r=r(\theta)} d\theta \end{aligned}$$

The contribution of the lower limit $r=\epsilon$ vanishes in the limit $\epsilon=0$. The weight functions considered in this appendix are admissible if they are applied to residuals which are sufficiently smooth.

Let P denote the weighted residual which one obtains by applying the weight function $-\psi_\xi$ to the potential equation (Eq. (5)).

$$P(x,y) = - \iint_{\Omega} \psi_\xi (\xi-x, \eta-y) \left[\frac{\partial}{\partial \xi} (\phi_\xi - (1/2)\phi_\xi^2) + \frac{\partial}{\partial \eta} (\phi_\eta) \right] d\xi d\eta \quad (A20)$$

The region Ω is the same as before (Fig. 2). The weight function $-\psi_\xi$ is regular at the shock (provided, of course, that the point (x,y) does not lie at the shock). The potential equation is supposed to be satisfied in the weak sense, in particular at the shock. The shock region is therefore included in the region Ω and no discussion of the shock conditions is needed. Carrying out the familiar integrations by parts one obtains

$$\begin{aligned} P(x,y) = & - \int_{\partial\Omega} \psi_\xi (\xi-x, \eta-y) [(\phi_\xi (\xi,\eta) - (1/2)\phi_\xi^2 (\xi,\eta)) d\eta - \phi_\eta (\xi,\eta) d\xi] \\ & + \int_{\partial\Omega} \phi (\xi,\eta) [\psi_{\xi\xi} (\xi-x, \eta-y) d\eta - \psi_{\xi\eta} (\xi-x, \eta-y) d\xi] \\ & - (1/2) \iint \psi_{\xi\xi} (\xi-x, \eta-y) \phi_\xi^2 (\xi,\eta) d\xi d\eta \end{aligned} \quad (A21)$$

With the use of the definition Eq. (A9), the second integral is transformed as follows

$$\begin{aligned} & \int_{\partial\Omega} \phi (\xi,\eta) [\psi_{\xi\xi} (\xi-x, \eta-y) d\eta - \psi_{\xi\eta} (\xi-x, \eta-y) d\xi] \\ = & - \int_{\partial\Omega} \phi (\xi,\eta) [\psi_{\eta\eta} (\xi-x, \eta-y) d\eta + \psi_{\xi\eta} (\xi-x, \eta-y) d\xi] \quad (A22) \\ = & - \int_{\partial\Omega} \phi (\xi,\eta) \frac{D}{D\eta} (\psi_\eta) d\eta = -\phi (\xi,\eta) \psi_\eta \Big|_{\text{limits}} + \int_{\partial\Omega} \psi_\eta (\xi-x, \eta-y) (\phi_\eta d\eta + \phi_\xi d\xi) \end{aligned}$$

The portions of the contour $\partial\Omega_1$, $\partial\Omega_2$ and $\partial\Omega_3$ form a closed curve. The contributions at the limits therefore vanish. The same holds for the small circle $\partial\Omega_4$. Across $\partial\Omega_3$ (portion of the wake), $\phi_\xi = u$ and $\phi_\eta = v$ are continuous, the integral therefore vanishes

for $\partial\Omega_3$. The first two integrals in Eq. (A21) evaluated for $\partial\Omega_2$ therefore give

$$\begin{aligned} & \int_{\partial\Omega_2} -[u(\xi, \eta) - (\frac{1}{2})u^2(\xi, \eta)]\psi_\xi(\xi-x, \eta-y)d\eta + u(\xi, \eta)\psi_\eta(\xi-x, \eta-y)d\xi \\ & + \int_{\partial\Omega_2} v(\xi, \eta)[\psi_\xi(\xi-x, \eta-y)d\xi + \psi_\eta(\xi-x, \eta-y)d\eta] \end{aligned} \quad (A23)$$

The contour $\partial\Omega_4$ (small circle around the point $(\xi, \eta) = (x, y)$) yields

$$\begin{aligned} & - \int_{\partial\Omega_4} \psi_\xi(\xi-x, \eta-y)[(\phi_\xi(\xi, \eta) - (\frac{1}{2})\phi_\xi^2(\xi, \eta))d\eta - \phi_\eta(\xi, \eta)d\xi] \\ & = -\lim_{\epsilon \rightarrow 0} \int_{\theta=0}^{2\pi} \int_{r=\epsilon} r^{-1} \cos\theta [u(\xi, \eta) - (\frac{1}{2})u^2(\xi, \eta)r\cos\theta + v(\xi, \eta)r\sin\theta]d\theta \\ & = \pi [u(x, y) - (\frac{1}{2})u^2(x, y)] \end{aligned} \quad (A24)$$

Evaluating Eq. (A22) for $\partial\Omega_4$ one obtains

$$\begin{aligned} & \int_{\partial\Omega_4} \psi_\eta(\xi-x, \eta-y)[\phi_\eta d\eta + \phi_\xi d\xi] \\ & = \lim_{\epsilon \rightarrow 0} \int_{\theta=0}^{2\pi} \int_{r=\epsilon} r^{-1} \sin\theta [v(\xi, \eta)r\cos\theta - u(\xi, \eta)r\sin\theta]d\theta \\ & = \pi u(x, y) \end{aligned} \quad (A25)$$

Substituting Eqs. (A23), (A24) and (A25) into Eq. (A21) and comparing with Eq. (A17) one finds that, indeed,

$$P(x, y) = Q_x(x, y) \quad (A26)$$

APPENDIX V

DERIVATION OF THE EXPRESSION Q_x BY APPLICATION OF THE
WEIGHT FUNCTION - ψ_ξ TO THE FORMULATION IN TERMS OF u AND v

The derivation to be shown here has of course great similarity to the one of Appendix IV. There are some differences in the details. One additional question arises. In the formulation of the problem in terms of the potential ϕ this quantity is always assumed to be continuous through the shock. To ensure this continuity in a computation in terms of u and v one must add the postulate Eq. (15), i.e.

$$[u]_-^+ (dx_s/dy) + [v]^+ = 0 \quad (A27)$$

A second shock condition is given by conservation of mass, Eq. (A14)

$$[u - (1/2)u^2]_-^+ - [v]^+ (dx_s/dy) = 0 \quad (A28)$$

First we show that the partial differential equations Eqs. (12) and (13) viz

$$\frac{\partial}{\partial x}(u - (1/2)u^2) + \frac{\partial}{\partial y}v = 0 \quad (A29)$$

$$u_y - v_x = 0 \quad (A30)$$

and the shock conditions (A27) and (A28) give the expression Q_x if one applies the weight function - ψ_ξ to the equation of conservation of mass (A29). Subsequently we shall ask whether the postulate $Q_x(x,y) = 0$ (for all values of (x,y)) ensures that Eqs. (A27) through (A29) are satisfied. It is assumed that the u field and the v field are always connected by Eq. (A30).

To derive the first result we apply an integration by parts to the following expression, in which the integrand is formed with Eq. (A29)

$$\begin{aligned}
P(x,y) &= - \iint_{\Omega} \psi_{\xi}(\xi-x, \eta-y) \left[\frac{\partial}{\partial \xi} (u(\xi, \eta) - (1/2)u^2(\xi, \eta)) + \frac{\partial}{\partial \eta} v \right] d\xi d\eta \\
&= - \int_{\partial\Omega} \psi_{\xi}(\xi-x, \eta-y) [(u(\xi, \eta) - (1/2)u^2(\xi, \eta)) d\eta - v(\xi, \eta) d\xi] \\
&\quad + \iint_{\Omega} [\psi_{\xi\xi}(\xi-x, \eta-y) (u(\xi, \eta) - (1/2)u^2(\xi, \eta)) + \psi_{\xi\eta}(\xi-x, \eta-y) v(\xi, \eta)] d\xi d\eta
\end{aligned}$$

One has

$$\begin{aligned}
&\iint_{\Omega} [\psi_{\xi\xi}(\xi-x, \eta-y) u(\xi, \eta) + \psi_{\xi\eta}(\xi-x, \eta-y) v(\xi, \eta)] d\xi d\eta \\
&= \iint_{\Omega} [-\psi_{\eta\eta}(\xi-x, \eta-y) u(\xi, \eta) + \psi_{\xi\eta}(\xi-x, \eta-y) v(\xi, \eta)] d\xi d\eta \\
&= \int_{\partial\Omega} \psi_{\eta}(\xi-x, \eta-y) u(\xi, \eta) d\xi + \psi_{\eta}(\xi-x, \eta-y) v(\xi, \eta) d\eta \\
&\quad + \iint_{\Omega} \psi_{\eta}(\xi-x, \eta-y) (u_{\eta}(\xi, \eta) - v_{\xi}(\xi, \eta)) d\xi d\eta
\end{aligned}$$

The double integral vanishes because of Eq. (A30). One thus obtains

$$\begin{aligned}
P(x,y) &= \int_{\partial\Omega} -[u(\xi, \eta) - (1/2)u^2(\xi, \eta)] \psi_{\xi}(\xi-x, \eta-y) d\eta + u(\xi, \eta) \psi_{\eta}(\xi-x, \eta-y) d\xi \\
&\quad + \int_{\partial\Omega} v(\xi, \eta) [\psi_{\xi}(\xi-x, \eta-y) d\xi + \psi_{\eta}(\xi-x, \eta-y) d\eta] \\
&\quad - (1/2) \iint_{\Omega} \psi_{\xi\xi}(\xi-x, \eta-y) u^2(\xi, \eta) d\xi d\eta \tag{A31}
\end{aligned}$$

The contribution of the contour $\partial\Omega_1$ (large circle which moves to infinity) vanishes, so does the contribution of the integral along the wake (contour $\partial\Omega_3$) because of the continuity of u and v . The contour $\partial\Omega_4$ (small circle around the point $(\xi, \eta) = (x, y)$) gives

$$2\pi(u(x,y) - (1/4)u^2(x,y)).$$

With these results one shows that indeed

$$P = Q_x.$$

The shock need not be considered as part of the boundary of Ω because we use the concept of weak equality in Eq. (A29).

Nevertheless let us introduce a contour $\partial\Omega_5$ which is formed by the two borders of a cut extending along the shock. (Traveling along the contour one must leave the interior of Ω to the left.) One therefore travels in the positive η direction on the upstream side of the shock. This contour gives a contribution

$$\begin{aligned} & \int_{\partial\Omega_5^+} \psi_\xi(\xi-x, \eta-y) [-(u(\xi, \eta) - (1/2)u^2(\xi, \eta)) + v(\xi, \eta)(d\xi_S/d\eta)]_-^+ d\eta \\ & + \int_{\partial\Omega_5^+} \psi_\eta(\xi-x, \eta-y) [u(\xi, \eta)(d\xi_S/d\eta) + v(\xi, \eta)]_-^+ d\eta \end{aligned} \quad (A32)$$

where $\partial\Omega_5^+$ is the upstream portion of the contour $\partial\Omega_5$. The two integrals vanish because of Eqs. (A27) and (A28).

Next we assume that we have a u field and a v field connected to it by Eq. (A30), and that the expression $P(x,y)$ vanishes for all values of (x,y) . Each value of P for a given (x,y) arises by the use of a weight function $\psi_\xi(\xi-x, \eta-y)$. By an argument similar to that shown at the beginning of Section III one can form linear combinations of such weight functions which represent weight functions $w(x,y)$ with finite support. We choose for this purpose weight functions which do not straddle the shock. Then it follows that Eq. (A29) is satisfied everywhere (in the weak sense) in the region where the shock is excluded. Therefore the residual expression (A31) for the region Ω with the shock excluded vanishes identically. Let us denote this expression by $\tilde{P}(x,y)$. $\tilde{P}(x,y)$ is identical with $P(x,y)$

except that the contribution of the shock boundary must be taken into account

$$\begin{aligned}\tilde{P}(x,y) = & P(x,y) + \int_{\partial\Omega_5} \psi_{\xi}(\xi-x, \eta-y) [-(u(\xi, \eta) - (1/2)u^2(\xi, \eta)) \\ & + v(\xi, \eta) (d\xi_S/d\eta)]_+^+ d\eta \\ & + \int_{\partial\Omega_5} \psi_{\eta}(\xi-x, \eta-y) [u(\xi, \eta) (d\xi_S/d\eta) + v(\xi, \eta)]_+^+ d\eta\end{aligned}$$

We have postulated that $P(x,y) = 0$ and we have shown that $\tilde{P}(x,y) = 0$.

The sum of the two integrals therefore vanishes identically in (x,y) . The sum of the two integrals represents solutions of the Laplace equation, in the whole x,y plane except at $\partial\Omega_5$ for $\psi_{\xi}(\xi-x, \eta-y)$ and $\psi_{\eta}(\xi-x, \eta-y)$ are solutions of the Laplace equation. At the shock this function has jumps in the normal and in the tangential derivatives which can be expressed in terms of the factors of ψ_{ξ} and ψ_{η} . But the above mentioned analytic function is identically equal to zero. The jumps are therefore zero and these two factors will vanish. Hence it follows that Eqs. (A27) and (A28) are satisfied.

APPENDIX VI
DERIVATION OF THE EXPRESSION Q_x FROM THE
PARTIAL DIFFERENTIAL EQUATION FOR u

In order to recognize the effect of the nonlinear term we introduce

$$q = \left(\frac{1}{2}\right) \frac{\partial}{\partial x} (u^2) \quad (A33)$$

This quantity can be regarded as a source term in the Laplace equation. The formulation of the problem in terms of u is given by Eqs. (15), (16), and (17). Then one has

$$u_{xx} + u_{yy} - q_x = 0 \quad (A34)$$

$$[u]_-^+ - [u^2]_-^+ (\cos^2 \beta) / 2 = 0 \quad (A35)$$

and as a substitute for Eq. (17) which is a consequence of Eq. (14)

$$-[u]_-^+ \operatorname{tg} \beta + [v]_-^+ = 0 \quad (A36)$$

We had, Eq. (7)

$$\operatorname{tg} \beta = -dx_g / dy$$

In Eq. (A36) it is assumed that one has constructed a v field, which is connected with the u field by the irrotationality condition.

$$u_y - v_x = 0 \quad (A37)$$

in the manner described at the end of Section II.

The difference between the present formulation and that of Appendix V lies in the fact that Eq. (A37) is used merely to determine the v field, while the basic problem is solved in terms of u . If there is a residual in Eq. (A34), then there will also be a

residual in the expression from which this equation is derived. Let the latter residual be $r(x,y)$. Accordingly

$$u_x + v_y - q = r(x,y) \quad (A38)$$

Then one obtains for the residual in Eq. (A34)

$$u_{xx} + u_{yy} - q_x = r_x \quad (A39)$$

The expression (A34) is not the equation of conservation of mass, but only its x derivative. In spite of its form it is not a conservation law, and therefore need not to be satisfied as one traverses a shock. Test functions of the kind used in the theory of generalized functions cannot be used if their support region straddles the shock. The functions ψ allows one to construct inadmissible as well as admissible test functions. We denote the residual expressions which arises from Eq. (A39) by \bar{P} .

$$\begin{aligned} \bar{P}(x,y) &= \iint_{\Omega} \psi(\xi-x, \eta-y) r_{\xi}(\xi, \eta) d\xi d\eta \\ &= \iint_{\Omega} \psi(\xi-x, \eta-y) (u_{\xi\xi}(\xi, \eta) + u_{\eta\eta}(\xi, \eta) - q_{\xi}(\xi, \eta)) d\xi d\eta \\ &= \int_{\partial\Omega} \psi(\xi-x, \eta-y) \{u_{\xi}(\xi, \eta) d\eta - u_{\eta}(\xi, \eta) d\xi\} \\ &\quad - \int_{\partial\Omega} u(\xi, \eta) \{\psi_{\xi}(\xi-x, \eta-y) d\eta - \psi_{\eta}(\xi-x, \eta-y) d\xi\} \\ &\quad - \iint_{\Omega} q_{\xi}(\xi, \eta) \psi(\xi-x, \eta-y) d\xi d\eta \end{aligned} \quad (A40)$$

Here the contour $\partial\Omega$ includes the two borders of the cut along the shock (denoted by $\partial\Omega_s$). In the first integral on the right we replace u by v by means of Eqs. (A37) and (A38)

$$\begin{aligned}
& \int_{\partial\Omega} \psi(\xi-x, \eta-y) \{u_{\xi}(\xi, \eta) d\eta - u_{\eta}(\xi, \eta) d\xi\} \\
&= - \int_{\partial\Omega} \psi(\xi-x, \eta-y) \{v_{\eta}(\xi, \eta) d\eta + v_{\xi}(\xi, \eta) d\xi\} + \int_{\partial\Omega} \psi(\xi-x, \eta-y) q(\xi, \eta) d\eta \\
&\quad + \int_{\partial\Omega} \psi(\xi-x, \eta-y) r(\xi, \eta) d\eta \\
&= \int_{\partial\Omega} v(\xi, \eta) \{\psi_{\xi}(\xi-x, \eta-y) d\xi + \psi_{\eta}(\xi-x, \eta-y) d\eta\} + \int_{\partial\Omega} \psi(\xi-x, \eta-y) q(\xi, \eta) d\eta \\
&\quad + \int_{\partial\Omega} \psi(\xi-x, \eta-y) r(\xi, \eta) d\eta
\end{aligned}$$

In the last step an integration by parts along the contour $\partial\Omega$ has been carried out. The term outside of the integral vanishes because the contour is closed. The integrals in Eq. (A40) over the contour at infinity $\partial\Omega_1$ vanish. So does the integral over the wake $\partial\Omega_3$ because of the continuity of u and v through the wake in a steady plane flow. The integral over the small circle around the point (x, y) is treated in a familiar manner and one obtains $2\pi u(x, y)$. One thus obtains from Eq. (A40)

$$\begin{aligned}
\bar{P}(x, y) &= 2\pi u(x, y) + \int_{\partial\Omega_2 + \partial\Omega_5} v(\xi, \eta) \{\psi_{\xi}(\xi-x, \eta-y) d\xi + \psi_{\eta}(\xi-x, \eta-y) d\eta\} \\
&\quad + \int_{\partial\Omega_2 + \partial\Omega_5} u(\xi, \eta) \{-\psi_{\xi}(\xi-x, \eta-y) d\eta + \psi_{\eta}(\xi-x, \eta-y) d\xi\} \\
&\quad + \int_{\partial\Omega_2 + \partial\Omega_5} \psi(\xi-x, \eta-y) q(\xi, \eta) d\eta \\
&\quad + \int_{\partial\Omega_2 + \partial\Omega_5} \psi(\xi-x, \eta-y) r(\xi, \eta) d\eta \\
&\quad - \iint_{\Omega} q_{\xi}(\xi, \eta) \psi(\xi-x, \eta-y) d\xi d\eta
\end{aligned} \tag{A41}$$

The function q which is the expression for the nonlinear terms enters the last equation, once because of the source terms in Eq. (A34), a second time when the derivatives of u are replaced by the derivatives of v (by means of Eqs. (A37) and (A38)). In the original differential equation the nonlinear terms appear in the form $q_x = -(1/2)\partial^2(u^2)/\partial x^2$; they are independent of the local value of u or its first derivative. The rewriting of the u derivatives generates a second nonlinear term. The derivation has been carried out in the form just shown to make clear that the term so obtained is not solely the source term in Eq. (A34). These terms can be combined

$$\begin{aligned} & -\iint_{\Omega} q(\xi, \eta) \psi(\xi-x, \eta-y) d\xi d\eta + \int_{\partial\Omega} q(\xi, \eta) \psi(\xi-x, \eta-y) d\xi \\ & = \iint_{\Omega} q(\xi, \eta) \psi_{\xi}(\xi-x, \eta-y) d\xi d\eta \\ & = (1/2) \iint_{\Omega} \frac{\partial}{\partial \xi} (u^2(\xi, \eta) \psi_{\xi}(\xi-x, \eta-y)) d\xi d\eta \end{aligned}$$

This expression can be transformed by a further integration by parts, so that derivatives of the unknown function u^2 no longer appear. In evaluating the result one must remember that one deals with the region Ω punctured at the point (x, y) . One obtains

$$\begin{aligned} & + (1/2) \iint_{\Omega} \frac{\partial}{\partial \xi} u^2(\xi, \eta) \psi_{\xi}(\xi-x, \eta-y) d\xi d\eta \\ & = (1/2) \int_{\partial\Omega} u^2(\xi, \eta) \psi_{\xi}(\xi-x, \eta-y) d\eta - (1/2) \iint_{\Omega} u^2(\xi, \eta) \psi_{\xi\xi}(\xi-x, \eta-y) d\xi d\eta \end{aligned}$$

Here the double integral must be evaluated for the punctured region Ω (if the point (x, y) lies within Ω). The last form has the advantage that it automatically takes δ functions into account which might for instance arise on the left side of the last expression if in some discretization procedure u^2 is approximated regionwise by constants.

The discussion just carried out has the purpose to explain that in the final form of the expression the nonlinear terms enter

in a rather complicated manner. The final expression for \bar{P} is best obtained by substituting immediately the form of q .

$$\begin{aligned}
 \bar{P}(x,y) &= \iint_{\Omega} \psi(\xi-x, \eta-y) \left\{ \frac{\partial^2}{\partial \xi^2} (u(\xi, \eta) - (1/2)u^2(\xi, \eta)) + u_{\eta\eta}(\xi, \eta) \right\} d\xi d\eta \\
 &= \int_{\partial\Omega} \psi(\xi-x, \eta-y) \left\{ \frac{\partial}{\partial \xi} (u(\xi, \eta) - (1/2)u^2(\xi, \eta)) d\eta - u_{\eta}(\xi, \eta) d\xi \right\} \\
 &\quad - \int_{\partial\Omega} \{ (u(\xi, \eta) - (1/2)u^2(\xi, \eta)) (\psi_{\xi}(\xi-x, \eta-y) d\eta - u(\xi, \eta) \psi_{\eta}(\xi-x, \eta-y) d\xi) \\
 &\quad - (1/2) \iint_{\Omega} \psi_{\xi\xi}(\xi-x, \eta-y) u^2(\xi, \eta) d\xi d\eta
 \end{aligned}
 \tag{A42}$$

In the first integral on the right, u is replaced by v by means of Eqs. (A37) and (A38). (A corresponding step has been carried out before.) One obtains

$$\begin{aligned}
 &- \int_{\partial\Omega} \psi(\xi-x, \eta-y) \{ v_{\eta}(\xi, \eta) d\eta + v_{\xi}(\xi, \eta) d\xi \} + \int_{\partial\Omega} \psi(\xi-x, \eta-y) r(\xi, \eta) d\xi \\
 &= \int_{\partial\Omega} v(\xi, \eta) \{ \psi_{\xi}(\xi-x, \eta-y) d\xi + \psi_{\eta}(\xi-x, \eta-y) d\eta \} + \int_{\partial\Omega} \psi(\xi-x, \eta-y) r(\xi, \eta) d\xi
 \end{aligned}$$

Again the terms outside of the integral vanish because the contour $\partial\Omega$ is closed. The integral for the small circle around the singular point (x,y) is treated as above. One obtains

$$\begin{aligned}
 \bar{P}(x,y) &= \alpha [\pi(u(x,y) - (1/4)u^2(x,y))] \\
 &\quad - \int_{\partial\Omega_2 + \partial\Omega_5} \psi_{\xi}(\xi-x, \eta-y) [(u(\xi, \eta) - (1/2)u^2(\xi, \eta)) d\eta - v(\xi, \eta) d\xi] \\
 &\quad + \int_{\partial\Omega_2 + \partial\Omega_5} \psi_{\eta}(\xi-x, \eta-y) [u(\xi, \eta) d\xi + v(\xi, \eta) d\eta]
 \end{aligned}
 \tag{A43}$$

(Continued
next page.)

$$- (1/2) \iint_{\Omega} \psi_{\xi\xi}(\xi-x, \eta-y) u^2(\xi, \eta) d\xi d\eta + \int_{\partial\Omega} \psi(\xi-x, \eta-y) r(\xi, \eta) d\eta \quad (A43)$$

(Concluded)

with $\alpha = 2$ for (x, y) within Ω
 $\alpha = 0$ for (x, y) outside Ω .

The contributions of the shock contour $\partial\Omega_5$ vanish because of the shock conditions (A35) and (A36). Therefore, by comparing with Eq. (A17) and using Eq. (A26)

$$P(x, y) = Q_x(x, y) = \bar{P}(x, y) - \int_{\partial\Omega} \psi(\xi-x, \eta-y) r(\xi, \eta) d\eta$$

This result is to be expected on the basis of the following reasoning. Using the original definitions one has, from Eqs. (A20) and (A38)

$$\begin{aligned} \bar{P}(x, y) &= \iint_{\Omega} \psi(\xi-x, \eta-y) r_{\xi}(\xi, \eta) d\xi d\eta \\ &= \int_{\partial\Omega} \psi(\xi, \eta) r(\xi, \eta) d\eta - \iint_{\Omega} r(\xi, \eta) \psi_{\xi}(\xi-x, \eta-y) d\xi d\eta \\ &= \int_{\partial\Omega} \psi(\xi-x, \eta-y) r(\xi, \eta) d\eta + P(x, y) \end{aligned}$$

The residual expressions \bar{P} (arising from a formulation in terms of u) and P (arising from a formulation in terms of ϕ or in terms of u and v) are closely related to each other but not identical. This is caused by the fact that Eq. (A39) with $r_x = 0$ merely guarantees that the function r which appears in Eq. (A38) is a function of y only. In working with Eq. (A39) we impose the additional requirement $r = 0$ at infinity and use the original expression I.

To treat the limiting case, in which the point (x, y) approaches the contour $\partial\Omega_2$ from the interior of Ω we rewrite the integrals over $\partial\Omega_2$ in Eq. (A43) as follows.

$$\begin{aligned}
& \int_{\partial\Omega_2} u(\xi, \eta) \{-\psi_\xi(\xi-x, \eta-y) d\eta + \psi_\eta(\xi-x, \eta-y) d\xi\} \\
& + \int_{\partial\Omega_2} v(\xi, \eta) \{\psi_\xi(\xi-x, \eta-y) d\xi + \psi_\eta(\xi-x, \eta-y) d\eta\} \\
& + (1/2) \int_{\partial\Omega_2} u^2(\xi, \eta) \psi_\xi(\xi-x, \eta-y) d\eta
\end{aligned} \tag{A44}$$

Let β be the angle of the inner normal of the region Ω with the x axis (Fig. 5). Then, for the unit vector, in the direction of the inner normal

$$e_n = e_x \cos \beta + e_y \sin \beta$$

and for the unit vector in the direction of the tangent to pointing in the direction in which the integration proceeds

$$e_s = e_x \sin \beta - e_y \cos \beta.$$

If $d\sigma$ is the line element of $\partial\Omega$ one has

$$\frac{d\xi}{d\sigma} = \sin \beta, \quad \frac{d\eta}{d\sigma} = -\cos \beta$$

then

$$\frac{d\psi}{dn} = \psi_\xi \cos \beta + \psi_\eta \sin \beta = -\psi_\xi \frac{d\eta}{d\sigma} + \psi_\eta \frac{d\xi}{d\sigma} \tag{A45}$$

$$\frac{d\psi}{dt} = \psi_\xi \sin \beta - \psi_\eta \cos \beta = \psi_\xi \frac{d\xi}{d\sigma} + \psi_\eta \frac{d\eta}{d\sigma}$$

$$\psi_\xi = \frac{d\psi}{dn} \cos \beta + \frac{d\psi}{dt} \sin \beta \tag{A46}$$

$$\psi_\xi \frac{d\eta}{d\sigma} = \frac{-d\psi}{dn} \cos^2 \beta - \frac{d\psi}{dt} \sin \beta \cos \beta.$$

The expression (A44) then assumes the form

$$\begin{aligned} & \int_{\partial\Omega_2} [u(\xi, \eta) - (1/2)\cos^2\beta(\sigma)u^2(\xi, \eta)] (d\psi/dn) d\sigma \\ & + \int_{\partial\Omega_2} [v(\xi, \eta) - (1/2)\sin\beta(\sigma)\cos\beta(\sigma)u^2(\xi, \eta)] (d\psi/dt) d\sigma \end{aligned} \quad (A47)$$

We defined

$$\psi = \log r(\xi-x, \eta-y).$$

We consider the limiting process where the point (x, y) approaches a point (x_c, y_c) along the inner normal. Applying a procedure familiar from potential theory, one treats the integral along $\partial\Omega_2$ in a small vicinity of the point (x_c, y_c) separately. In the limit where (x, y) coincides with x_c, y_c the integrand of the first integral in Eq. (A47) remains bounded in the outer region (where the point (x_c, y_c) is excluded. For the small region which contains the point (x_c, y_c) one obtains

$$\int (d\psi/dn) d\sigma = -\pi.$$

The second integral in Eq. (A47) must be evaluated as Cauchy principal value.

One thus obtains, from Eq. (A17), for the point $(x(s), y(s))$ on the contour

$$\begin{aligned} P(x, y) = & \pi(u(x, y) - (1/2)\sin^2\beta(s)u^2(x, y)) \\ & + \int_{\partial\Omega_2} \{u(\xi, \eta) - (1/2)\cos^2\beta(s)u^2(\xi, \eta)\} \{-\psi_\xi(\xi-x, \eta-y) d\eta + \psi_\eta(\xi-x, \eta-y) d\xi\} \\ & + \int_{\partial\Omega_2} \{v(\xi, \eta) - (1/2)\sin\beta(s)\cos\beta(s)u^2(\xi, \eta)\} \{\psi_\xi(\xi-x, \eta-y) d\xi + \psi_\eta(\xi-x, \eta-y) d\eta\} \\ & - (1/2) \iint_{\Omega} \psi_{\xi\xi}(\xi-x, \eta-y) u^2(\xi, \eta) d\xi d\eta \end{aligned} \quad (A48)$$

To avoid an ambiguity in the signs, ψ_ξ and ψ_η , instead of ψ_n and ψ_t have been reintroduced, although the ψ_n and ψ_t express the nature of the integrands more clearly.

The evaluation of the double integral requires some precautions because of the singularity introduced by the factor $\psi_{\xi\xi}$ in the integrand. The region under consideration is the punctured region Ω . In a specific case the procedure depends upon the chosen representation for u^2 . To show what will happen we consider the case where u^2 is constant within an element bounded by straight lines. Then one obtains

$$-(1/2)u^2 \iint_{\Omega'} \psi_{\xi\xi} d\xi d\eta = -(1/2)u^2 \int_{\partial\Omega'} \psi_\xi d\eta$$

where Ω' is the element under consideration and $\partial\Omega'$ its boundary. If the point (x,y) lies within the element, then for the small circle around this point

$$\int \psi_\xi d\eta = -\pi.$$

For the remainder of the contour it does not matter whether the point (x,y) lies inside or outside of the contour.

We introduce $\xi-x = \xi'$, $\eta-y = \eta'$. Then one has

$$\psi_\xi = \frac{\xi'}{\xi'^2 + \eta'^2}$$

Let the contour between the points A and B in Fig. 6 be given by

$$\xi' = a - \eta' \operatorname{tg} \beta$$

where β is the angle between the inner normal to the boundary AP and the x axis (Fig. 6.) Then one has to evaluate

$$\int_{\eta_A}^{\eta_B} \psi_{\xi'} d\eta' = \int \frac{\xi'^2 d\eta'}{\xi'^2 + \eta'^2} = \int_{\eta_A}^{\eta_B} \frac{a - \operatorname{tg} \beta \eta'}{(a - \eta' \operatorname{tg} \beta)^2 + \eta'^2} d\eta'$$

The direction of integration is chosen in such a manner that the interior is to the left as one travels along the contour. This is an elementary integral.

$$\begin{aligned} \int_{\eta_A}^{\eta_B} \frac{a - \operatorname{tg} \beta \eta'}{(a - \eta' \operatorname{tg} \beta)^2 + \eta'^2} d\eta' &= -\sin \beta \cos \beta \int_{\eta_A}^{\eta_B} \frac{(\frac{\eta'}{\cos \beta} - a \sin \beta) (\frac{d\eta'}{\cos \beta})}{(\frac{\eta'}{\cos \beta} - a \sin \beta)^2 + a^2 \cos^2 \beta} \\ &+ \cos^2 \beta \int_{\eta_A}^{\eta_B} \frac{\frac{d\eta'}{a \cos^2 \beta}}{[(\frac{\eta'}{\cos \beta} - a \sin \beta) / a \cos \beta]^2 + 1} \\ &= -\sin \beta \cos \beta (1/2) \log \left[\left(\frac{\eta}{\cos \beta} - a \sin \beta \right)^2 + a^2 \cos^2 \beta \right] \Big|_{\eta_A}^{\eta_B} \\ &+ \cos^2 \beta \operatorname{arctg} \frac{\frac{\eta'}{\cos \beta} - a \sin \beta}{a \cos \beta} \Big|_{\eta_A}^{\eta_B} \end{aligned}$$

One has, either from Fig. 7 or by direct computation,

$$\begin{aligned} r_A^2 &= \eta'^2 + \xi'^2 = \left(\frac{\eta'}{\cos \beta} - a \sin \beta \right)^2 + a^2 \cos^2 \beta \\ \theta_A &= \operatorname{arctg} \frac{\frac{\eta'}{\cos \beta} - a \sin \beta}{a \cos \beta} \end{aligned}$$

Thus, finally as the contribution of the boundary AB

$$-(1/2)u^2 [-\sin \beta \cos \beta \log \frac{r_B}{r_A} + \cos^2 \beta (\theta_B - \theta_A)]$$

We apply this result to several special cases.

Let the point (x,y) lie in the middle of a square (Fig. 8a). One then obtains

$$-(1/2)u^2 \int_{\Omega} \psi_{\xi\xi} d\xi d\eta = -\frac{1}{2} u^2 [-\pi + \pi] = 0$$

The first factor comes from the contour around the point (x,y) . The second one from the contour AB and CD. Incidentally, one sees that the exclusion of the point (x,y) by a square is equivalent to the exclusion by a small circle.

For a narrow rectangle (extending in the y direction, Fig. 8b) one obtains

$$(-1/2)u^2 \int_{\Omega} \psi_{\xi\xi} d\xi d\eta = -\frac{1}{2} (-\pi + 2\pi) = -\frac{\pi}{2}.$$

If the point (x,y) approaches the contour AB, Fig. 8c, then the contribution of the contour AB cancels the contribution of the small circle. In this manner one can discuss the problem and contours of different kind; for instance, the contribution of an oblique contour.

One finds that the local contribution of the double integral in Eq. (A47) will cancel the local contribution $(-1/2)\sin^2\beta u(x,y)$ and also the local contributions of the second contour integral (in which the Cauchy principal value is to be formed).

APPENDIX VII

RELATION BETWEEN THE SHOCK CONDITIONS EQS. (14) AND (15) AND LOCAL EXPRESSIONS Q_x or Q_y

It has already been shown in Appendix V that the shock conditions (14) and (16) are satisfied if $Q_x(x,y) = 0$ identically in x and y . In this appendix it will be shown that if $Q_x = 0$ or $Q_y = 0$ locally at points immediately upstream and downstream of the shock then, respectively, Eq. (14) and a linear combination of Eqs. (14) and (16) will be satisfied. In other words, if one uses in the computation either conditions for Q_x or Q_y then one relation derived from the two shock conditions is satisfied even if $Q_x = 0$ or $Q_y = 0$ only at points immediately upstream and downstream of the shock.

This discussion is primarily of interest for the classical use of the integral equation method. Accordingly, it suffices if we restrict it to the specialized equations (25) and (26) although the results have general validity. Let us first consider the condition derived from Q_x . We evaluate the expression (26) for points (x,y) approaching the shock from the upstream and the downstream side along a line $y = \text{const}$. Let the point of the shock reached in the limit be $x_s(y)$. A discontinuity in the expression Q_x can be caused by the terms $2\pi u(x,y)$, $-(\pi/2) u^2(x,y)$ and

$$-(1/2) \iint_{\Omega} u^2(\xi, \eta) \psi_{\xi\xi}(\xi-x, \eta-y) d\xi d\eta$$

As always the region Ω is part of the ξ, η plane punctured at $(\xi, \eta) = (x, y)$. We enclose the point (x_s, y) by a small box (Fig. 9). In the limiting processes where one approaches this point from the upstream and downstream side the singular point (x, y) moves within the box. We consider the limits of the double integral for

$$x = \lim_{\epsilon \rightarrow 0} (x_s - \epsilon) \quad \text{and} \quad x = \lim_{\epsilon \rightarrow 0} (x_s + \epsilon); \quad \epsilon > 0$$

In this limiting process the contributions of the double integral from outside of the box will be continuous as x moves through the shock. Only the region within the box need to be considered. Let Ω^+ and Ω^- be the portions of Ω within the box upstream and downstream of the shock. Let u^+ and u^- be the values of u upstream and downstream of the shock. The difference of the double integral for the area $\Omega^+ + \Omega^-$ with x once approaching the shock from the upstream side and a second time from the downstream side is given by

$$\begin{aligned}
 D = & \iint_{\Omega^+} (u^+)^2(\xi, \eta) \psi_{\xi\xi}(\xi - x_s(y) + \epsilon, \eta - y) d\xi d\eta \\
 & + \iint_{\Omega^-} (u^-)^2(\xi, \eta) \psi_{\xi\xi}(\xi - x_s(y) + \epsilon, \eta - y) d\xi d\eta \\
 & - \iint_{\Omega^+} (u^+)^2(\xi, \eta) \psi_{\xi\xi}(\xi - x_s(y) - \epsilon, \eta - y) d\xi d\eta \\
 & - \iint_{\Omega^-} (u^-)^2(\xi, \eta) \psi_{\xi\xi}(\xi - x_s(y) - \epsilon, \eta - y) d\xi d\eta
 \end{aligned}$$

In evaluating the first and fourth of these integrals one must take the presence of the singular points at $x_s - \epsilon, y$ and $x_s + \epsilon, y$ into account. One can take the box sufficiently small so that u^+ and u^- can be considered as constant. (one can indeed, show that higher order terms in the development of u do not contribute to a discontinuity. Then the integration with respect to ξ can be carried out in the above integrals and one obtains only integrals over the contour. The evaluation of the contours around the singular points is familiar. One obtains

$$-\pi(u^+)^2 \text{ and } -\pi(u^-)^2.$$

Therefore

$$\begin{aligned}
D = & -\pi(u^+)^2 + (u^+)^2 \int_{\partial\Omega^+} \psi_\xi(\xi-x_s(y) + \varepsilon, \eta-y) d\eta \\
& + (u^-)^2 \int_{\partial\Omega^-} \psi_\xi(\xi-x_s(y) + \varepsilon, \eta-y) d\eta \\
& - (u^+)^2 \int_{\partial\Omega^+} \psi_\xi(\xi-x_s(y) - \varepsilon, \eta-y) d\eta \\
& - (u^-)^2 \int_{\partial\Omega^-} \psi_\xi(\xi-x_s(y) - \varepsilon, \eta-y) d\eta + \pi(u^-)^2
\end{aligned}$$

Here $\partial\Omega^+$ and $\partial\Omega^-$ denote the contours of the areas Ω^+ and Ω^- , but with the exclusion of the contours around the singular points. These contours include the borders of the cut made along the shock. At the outer boundaries of the regions Ω^+ and Ω^- namely $\xi = x_1$, $y_1 < \eta < y_2$ and $\xi = x_2$; $y_1 < \eta < y_2$ (Fig. 9) one has obviously

$$\lim_{\varepsilon \rightarrow 0} \psi_\xi(\xi-x_s(y) + \varepsilon, \eta-y) = \lim_{\varepsilon \rightarrow 0} \psi_\xi(\xi-x_s(y) - \varepsilon, \eta-y)$$

In the limit $\varepsilon \rightarrow 0$ these parts of the contour do not contribute to the difference D ; only the integrals along the shock are left. One observes that this integration is carried out in opposite directions for the contours $\partial\Omega^+$ and $\partial\Omega^-$, on the upstream side in the direction of increasing η . One therefore obtains

$$\begin{aligned}
D = & [(u^+)^2 - (u^-)^2] \left\{ -\pi + \int_{y_1}^{y_2} \psi_\xi(\xi-x_s(y) + \varepsilon, \eta-y) d\eta \right. \\
& \left. - \int_{y_1}^{y_2} \psi_\xi(\xi-x_s(y) - \varepsilon, \eta-y) d\eta \right\}
\end{aligned}$$

One has

$$\psi_{\xi}(\xi-x, \eta-y) = \frac{(\xi-x)}{(\xi-x)^2 + (\eta-y)^2}$$

We set for the evaluation of the integrals

$$\xi-x_s(y) = \xi', \eta-y = \eta'$$

and for the limits of the integral

$$\eta'_1 = y_1 - y, \quad \eta'_2 = y_2 - y.$$

Notice that

$$\eta'_1 < 0 < \eta'_2$$

For the present purpose it suffices if one replaces the shock by a straight line

$$\xi' = -\eta' \operatorname{tg} \beta$$

where (as in Section 2) β is the local angle between the downstream pointing normal to the shock and the x axis. Then the first of the above integral assumes the form

$$I_1 = \int_{\eta'_1}^{\eta'_2} \frac{\xi' + \epsilon}{(\xi' + \epsilon)^2 + \eta'^2} d\eta' = \int_{\eta'_1}^{\eta'_2} \frac{-\operatorname{tg} \beta \eta' + \epsilon}{(-\operatorname{tg} \beta \eta' + \epsilon)^2 + \eta'^2} d\eta'$$

We show the essential steps in the evaluation of this elementary integral

$$(-\operatorname{tg} \beta \eta' + \varepsilon)^2 + \eta'^2 = \varepsilon^2 \cos^2 \beta \left[\left(\frac{\eta'}{\varepsilon \cos^2 \beta} - \operatorname{tg} \beta \right)^2 + 1 \right]$$

Setting

$$\frac{\eta'}{\varepsilon \cos^2 \beta} - \operatorname{tg} \beta = v$$

one obtains

$$I_1 = \int_{v_1}^{v_2} \frac{-\sin \beta \cos \beta v + \cos^2 \beta}{1+v^2} dv$$

where the limits of the integral are given by

$$v_1 = \frac{\eta'_1}{\varepsilon \cos^2 \beta} - \operatorname{tg} \beta \quad \text{and} \quad v_2 = \frac{\eta'_2}{\varepsilon \cos^2 \beta} - \operatorname{tg} \beta$$

For $\varepsilon \rightarrow 0$, $\varepsilon > 0$ one has $v_1 \rightarrow -\infty$, $v_2 \rightarrow +\infty$

The contribution of the first summand in the integrand of I_1 vanishes because of antisymmetry and one obtains

$$I_1 = \cos^2 \beta \pi$$

The second integral in the expression D is treated in the same manner, but ε is replaced by $-\varepsilon$. Therefore $v_1 = +\infty$ and $v_2 = -\infty$

One thus finds

$$D = ((u^+)^2 - (u^-)^2) (-\pi + 2\pi \cos^2 \beta)$$

Forming the difference of the expressions for Q_x , (Eq. 25) upstream and downstream of the shock in the limit $\varepsilon \rightarrow 0$, one obtains with the notation of Section 2

$$\begin{aligned} 2\pi[u]_-^+ - \pi/2[u^2]_-^+ - (1/2)D \\ = 2\pi[u]_-^+ - [u^2]_-^+ (\pi/2 - \pi/2 + \pi \cos^2 \beta) = 0 \end{aligned}$$

This is the condition Eq. (16).

An analogous procedure carried out in Eq. (26) requires the evaluation of

$$-(1/2) \iint_{\Omega} u^2(\xi, \eta) \psi_{\xi\eta}(\xi-x, \eta-y) d\xi d\eta$$

After an integration with respect to ξ one arrives again at integrals around the contours $\partial\Omega^+$ and $\partial\Omega^-$. No contribution due to the singular point (x, y) appears. One arrives at

$$\begin{aligned} -\frac{1}{2}[u^2]_-^+ \left\{ \int_{\eta_1}^{\eta_2} \psi_{\eta}(\xi-x_s(y)+\varepsilon, \eta-y) d\eta \right. \\ \left. - \int_{\eta_1}^{\eta_2} \psi_{\eta}(\xi-x_s(y)-s, \eta-y) d\eta \right\} \end{aligned}$$

Here

$$\psi_{\eta}(\xi-x, \eta-y) = \frac{\eta-y}{(\xi-x)^2 + (\eta-y)^2}$$

In the integral one makes the same substitution as before. One finally obtains from Eq. (26)

$$2[v]_-^+ + [u^2]_-^+ \sin\beta \cdot \cos\beta = 0$$

The conditions derived for, Q_y therefore lead to a linear combination of the shock conditions (14) and (16).

The full shock conditions are obtained only if simultaneously $Q_x = 0$ and $Q_y = 0$ for points immediately upstream and downstream of the shock. The application of only $Q_x = 0$ or $Q_y = 0$ yields the full shock conditions only if the respective expression vanishes globally.

APPENDIX VIII
DERIVATION PERTAINING TO EQUATIONS (35) AND (36)

The evaluation of the normal derivative of ϕ in Eq. (35) requires some caution because the singularity which arises in the derivatives of ψ as the point (x,y) approaches the contour $\partial\Omega$. Therefore, we show some details. Let $\beta(s)$ be the angle of the inner normal to the contour of the distant field with the x axis (Fig. 5). In the contour integrals the variable s will be replaced by an umbral variable σ . We assume that s (and σ) increase in a direction of travel for which the distant field is to the left. The contour is described by $x = x(s)$ and $y = y(s)$. We define unit vectors in the direction of the normal and in the tangential directions.

$$\begin{aligned} e_n &= e_x \cos\beta + e_y \sin\beta \\ e_t &= e_x \sin\beta - e_y \cos\beta \end{aligned}$$

Derivatives of ϕ in the direction of e_n and e_t are denoted by ϕ_n and ϕ_t

$$\begin{aligned} \phi_n(x(s), y(s)) &= \phi_x \cos\beta(s) + \phi_y \sin\beta(s) \\ \phi_t(x(s), y(s)) &= \phi_x \sin\beta(s) - \phi_y \cos\beta(s) \end{aligned}$$

We assume (x,y) to be fixed inside of the distant field Ω , and define derivatives ψ_n and ψ_t . The differentiations are then carried out with respect to ξ and η .

$$\begin{aligned} \psi_n(\sigma, x, y) &= \psi_n(\xi(\sigma) - x, \eta(\sigma) - y) = \\ &= \psi_\xi(\xi(\sigma) - x, \eta(\sigma) - y) \cos\beta(\sigma) + \psi_\eta(\xi(\sigma) - x, \eta(\sigma) - y) \sin\beta(\sigma) \\ \psi_t(\sigma, x, y) &= \psi_\xi \sin\beta(\sigma) - \psi_\eta \cos\beta(\sigma). \end{aligned} \tag{A49}$$

(arguments of ψ_ξ and ψ_η as above).

Then one has

$$\psi_{\xi}(\xi(\sigma)-x, \eta(\sigma)-y) = \psi_n \cos \beta(\sigma) + \psi_t \sin \beta(\sigma)$$

$$\psi_{\eta}(\xi(\sigma)-x, \eta(\sigma)-y) = \psi_n \sin \beta(\sigma) - \psi_t \cos \beta(\sigma)$$

Now from Eq. (34)

$$\begin{aligned} \phi_x(x, y) = (2\pi)^{-1} & \left[- \int_{\partial\Omega} f(\sigma) \psi_{\xi}(\xi(\sigma)-x, \eta(\sigma)-y) d\sigma - \iint_{\Omega} q(\xi, \eta) \psi_{\xi}(\xi-x, \eta-y) d\xi d\eta \right. \\ & \left. + \Gamma r(x, y)^{-1} \sin \theta(x, y) \right] \end{aligned} \quad (A50)$$

$$\begin{aligned} \phi_y(x, y) = (2\pi)^{-1} & \left[- \int_{\partial\Omega} f(\sigma) \psi_{\eta}(\xi(\sigma)-x, \eta(\sigma)-y) d\sigma - \iint_{\Omega} q(\xi, \eta) \psi_{\eta}(\xi-x, \eta-y) d\xi d\eta \right. \\ & \left. - \Gamma r(x, y)^{-1} \cos \theta(x, y) \right] \end{aligned} \quad (A51)$$

with $r^2 = x^2 + y^2$, $\sin \theta(x, y) = y/r$, $\cos \theta(x, y) = x/r$.

The above expressions for ψ_{ξ} and ψ_{η} are substituted in the integrals over $\partial\Omega$. We form

$$\begin{aligned} \phi_x(x, y) \cos \beta(s) + \phi_y(x, y) \sin \beta(s) = & (2\pi)^{-1} \left\{ - \int_{\partial\Omega} f(\sigma) \psi_n(\sigma, x, y) \cos(\beta(\sigma) - \beta(s)) d\sigma \right. \\ & - \int_{\partial\Omega} f(\sigma) \psi_t(\sigma, x, y) \sin(\beta(\sigma) - \beta(s)) d\sigma \\ & - \cos \beta(s) \iint_{\Omega} q(\xi, \eta) \psi_{\xi}(\xi-x, \eta-y) d\xi d\eta \\ & - \sin \beta(s) \iint_{\Omega} q(\xi, \eta) \psi_{\eta}(\xi-x, \eta-y) d\xi d\eta \\ & \left. - (\Gamma/r) \sin(\beta(s) - \theta(x, y)) \right\} \end{aligned}$$

and

$$\begin{aligned}
& \phi_x(x,y)\sin\beta(s) - \phi_y(x,y)\cos\beta(s) \\
&= (2\pi)^{-1} \int_{\partial\Omega} f(\sigma)\psi_n(\sigma,x,y)\sin(\beta(s)-\beta(\sigma))d\sigma \\
&\quad - \int_{\partial\Omega} f(\sigma)\psi_t(\sigma,x,y)(-\cos(\beta(s)-\beta(\sigma)))d\sigma \\
&\quad - \sin\beta(s) \iint_{\Omega} q(\xi,\eta)\psi_{\xi}(\xi-x,\eta-y)d\xi d\eta \\
&\quad + \cos\beta(s) \iint_{\Omega} q(\xi,\eta)\psi_{\eta}(\xi-x,\eta-y)d\xi d\eta \\
&\quad - (\Gamma/r)\cos(\beta(s) - \theta(x,y))
\end{aligned}$$

Now one carries out in the last formulae the limiting process where (x,y) approaches a point of the contour $(x(s),y(s))$. The results are denoted by $\phi_n(s)$ and $\phi_t(s)$

$$\begin{aligned}
\phi_n(s) &= (f(s)/2) + (2\pi)^{-1} \left\{ - \int_{\partial\Omega} f(\sigma)\psi_n(\sigma,(x(s),y(s)))\cos(\beta(\sigma)-\beta(s))d\sigma \right. \\
&\quad - \int_{\partial\Omega} f(\sigma)\psi_t(\sigma,x(s),y(s))\sin(\beta(\sigma)-\beta(s))d\sigma \\
&\quad - \cos\beta(s) \iint_{\Omega} q(\xi,\eta)\psi_{\xi}(\xi-x(s),\eta-y(s))d\xi d\eta \\
&\quad - \sin\beta(s) \iint_{\Omega} q(\xi,\eta)\psi_{\eta}(\xi-x(s),\eta-y(s))d\xi d\eta \\
&\quad \left. - (\Gamma/r)\sin(\beta(s) - \theta(x(s),y(s))) \right\}
\end{aligned} \tag{A52}$$

$$\begin{aligned}
\phi_t(s) &= (2\pi)^{-1} \left\{ - \int_{\partial\Omega} f(\sigma)\psi_n(\sigma,x(s),y(s))\sin(\beta(\sigma)-\beta(s))d\sigma \right. \\
&\quad + \int_{\partial\Omega} f(\sigma)\psi_t(\sigma,x(s),y(s))\cos(\beta(\sigma)-\beta(s))d\sigma \\
&\quad \left. - \sin\beta(s) \iint_{\Omega} q(\xi,\eta)\psi_{\xi}(\xi-x,\eta-y)d\xi d\eta \right\}
\end{aligned} \tag{A53}$$

(Continued on next page.)

$$+ \cos \beta(\sigma) \iint_{\Omega} q(\xi, \eta) \psi_{\eta}(\xi-x, \eta-y) d\xi d\eta \\ - (\Gamma/r) \cos(\beta(s) - \theta(x(\sigma), y(s))) \}$$

The present representation of $\phi_n(s)$ and $\phi_t(s)$, which uses ψ and ψ_t may appear less direct than a representation in ψ_{ξ} and ψ_{η} but it shows more clearly the limiting form of the equations. If the point $(\xi(\sigma), \eta(\sigma))$ approaches the point $(x(s), y(s))$ (in other words, as σ approaches s). The derivative ψ_t behaves as $(\sigma-s)^{-1}$; integrals containing ψ_t must be evaluated in the sense of Cauchy principal values (unless there is another factor in the integrand which tends to zero in the same manner). Furthermore,

$$\lim_{\sigma \rightarrow s} \psi_n = \psi_n(\sigma, x(s), y(s)) = -(2d\beta(s)/ds)^{-1}$$

(One will remember that s increases in a direction in which one has the distant field to the left. $d\beta/ds$ is positive if the distant field is convex at the point s .)

If one evaluates the double integrals in the form used in Eq. (35) then it does not matter whether the plane Ω is punctured or not. If, however, one transforms it by integrations by part so that the derivatives of u^2 in q vanishes, then one must remember that one deals with the punctured plane in the limit where the singular point approaches the boundary from the inside of the distant field.

$$\begin{aligned} \iint_{\Omega} q(\xi, \eta) \psi_{\xi}(\xi-x, \eta-y) d\xi d\eta &= (1/2) \iint_{\Omega} \frac{\partial}{\partial \xi} (u^2(\xi, \eta)) \psi_{\xi}(\xi-x, \eta-y) d\xi d\eta \\ &= (1/2) \int_{\partial \Omega} u^2(\xi, \eta) \psi_{\xi}(\xi-x, \eta-y) d\eta - (1/2) \iint_{\Omega} u^2(\xi, \eta) \psi_{\xi\xi}(\xi-x, \eta-y) d\xi d\eta \\ &= -(\pi/4) u^2(x, y) + (1/2) \int_{\partial \Omega_2} u^2(\xi(\sigma), \eta(\sigma)) \psi_{\xi}(\xi(\sigma)-x, \eta(\sigma)-y) d\eta(\sigma) \\ &\quad - (1/2) \iint_{\Omega} u^2(\xi, \eta) \psi_{\xi\xi}(\xi-x, \eta-y) d\xi d\eta \\ &\quad \lim_{\substack{x \rightarrow x(s) \\ y \rightarrow y(s)}} \end{aligned}$$

We have found above

$$\psi_{\xi}(\xi(\sigma)-x, \eta(\sigma)-y) = \psi_n \cos \beta(\sigma) + \psi_t \sin \beta(\sigma)$$

and $d\eta/d\sigma = -\cos \beta(\sigma)$. Then

$$\begin{aligned} & (1/2) \int_{\partial \Omega_2} u^2(\xi(\sigma), \eta(\sigma)) \psi_{\xi}(\xi(\sigma)-x, \eta(\sigma)-y) d\eta(\sigma) \\ &= (1/2) \int_{\partial \Omega_2} u^2(\xi(\sigma), \eta(\sigma)) [\psi_n(\sigma, x, y) \cos \beta(\sigma) \\ &+ \psi_t(\sigma, x, y) \sin \beta(\sigma)] (-\cos \beta(\sigma)) d\sigma = -(\pi/4) \cos^2 \beta(s) u^2(x(s), y(s)) \\ &- (1/2) \int_{\partial \Omega_2} u^2(\xi(\sigma), \eta(\sigma)) [\psi_n(\sigma, x(s), y(s)) \cos^2 \beta(\sigma) + \psi_t(\sigma, x(s), y(s)) \sin \beta(\sigma) \cos \beta(\sigma)] d\sigma \end{aligned}$$

The portion of the integral involving ψ_t is evaluated as Cauchy principal value. For the evaluation of the double integral, see Appendix VI.

In a corresponding manner one obtains

$$\begin{aligned} \iint_{\Omega} q(\xi, \eta) \psi_{\eta}(\xi-x, \eta-y) d\xi d\eta &= \frac{1}{2} \iint_{\partial \Omega} \frac{\partial}{\partial \xi} u^2(\xi, \eta) \psi_{\eta}(\xi-x, \eta-y) d\xi d\eta \\ &= \frac{1}{2} \int_{\partial \Omega} u^2(\xi, \eta) \psi_{\eta}(\xi-x, \eta-y) d\eta - (1/2) \iint_{\Omega} u^2(\xi, \eta) \psi_{\xi\eta}(\xi-x, \eta-y) d\xi d\eta \\ &= \frac{1}{2} \int_{\partial \Omega_2} u^2(\xi, \eta) (\psi_n \sin \beta(\sigma) - \psi_t \cos \beta(\sigma)) (-\cos \beta(\sigma)) d\sigma \\ &- (1/2) \iint_{\Omega} u^2(\xi, \eta) \psi_{\xi\eta}(\xi-x, \eta-y) d\xi d\eta \\ &= +(\pi/4) u^2(x(s), y(s)) \sin \beta(s) \cos \beta(s) \\ &+ \frac{1}{2} \int_{\partial \Omega_2} u^2(\xi(\sigma), \eta(\sigma)) [-\psi_n(\sigma, x(s), y(s)) \sin \beta(\sigma) \cos \beta(\sigma) \\ &+ \psi_t(\sigma, x(s), y(s)) \cos^2 \beta(\sigma)] d\sigma \\ &- (1/2) \iint_{\Omega} u^2(\xi, \eta) \psi_{\xi\eta}(\xi-x(s), \eta-y(s)) d\xi d\eta \\ \lim_{\substack{x \rightarrow x(s) \\ y \rightarrow y(s)}} & \end{aligned}$$

In practice one will use a contour $\partial\Omega_2$ which consists of straight lines oriented in the x and y directions and considerable simplifications will be encountered, because $\cos\beta$ and $\sin\beta$ will then be either zero or one.

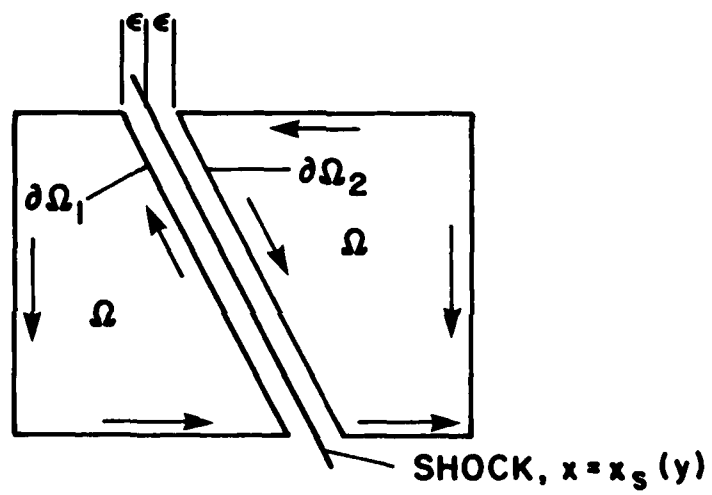


Figure 1. Region Ω in the Vicinity of a Shock.

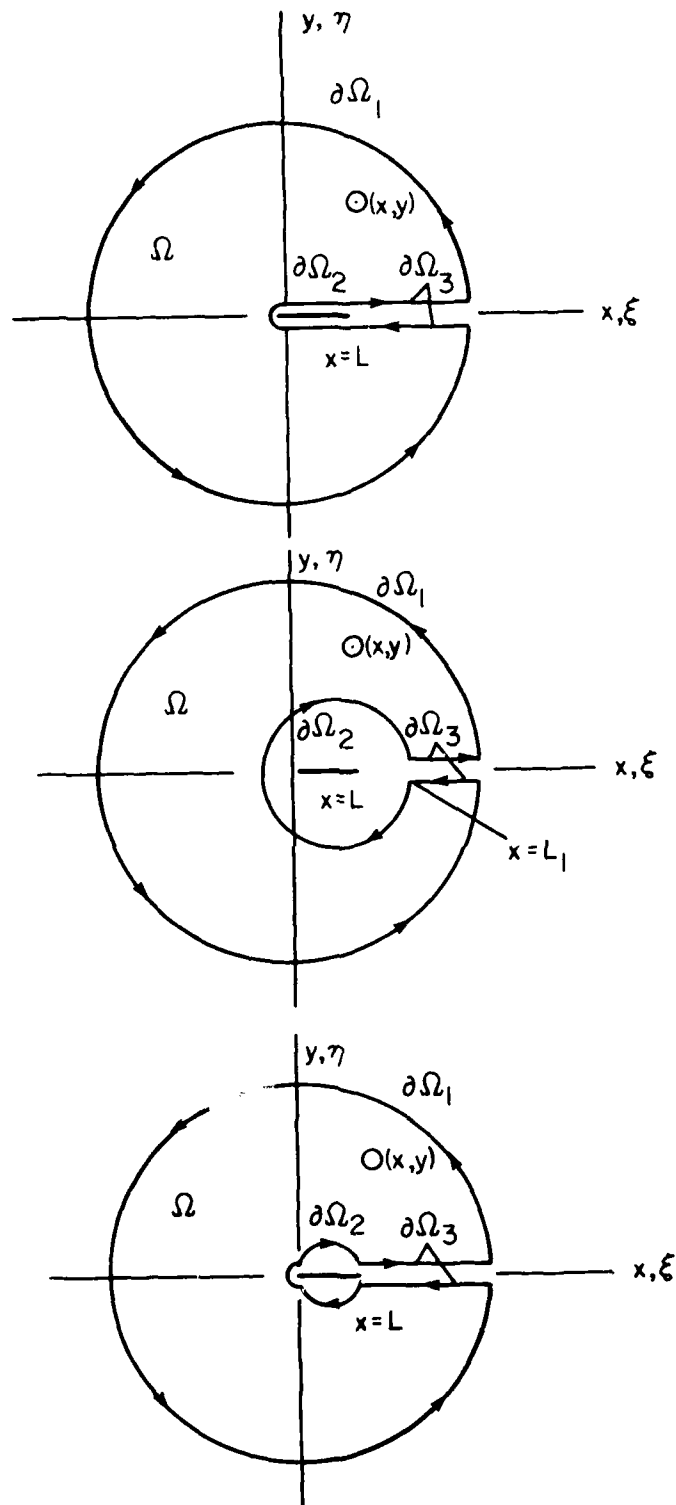


Figure 2. Different Forms of the Boundaries of the Region Ω .

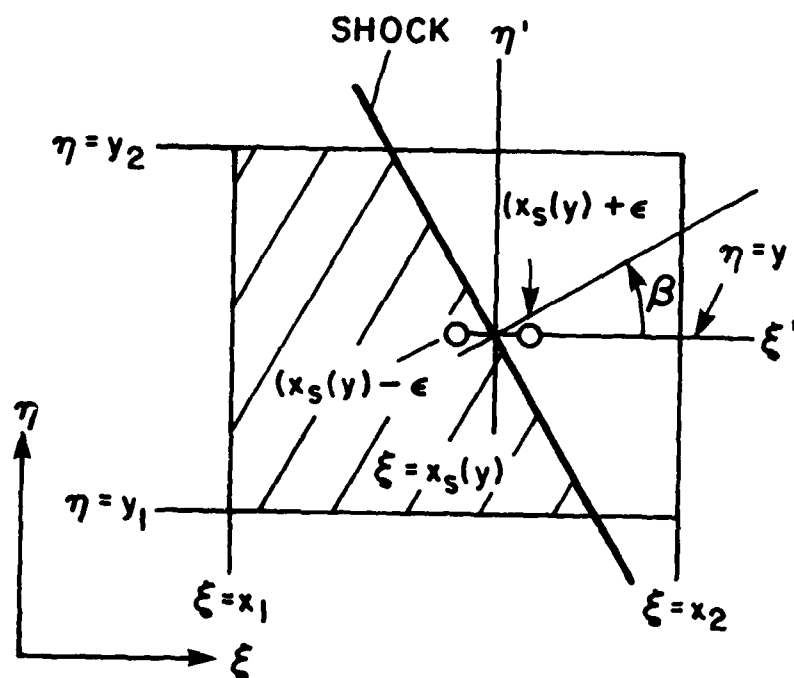


Figure 3. Notation Used in Demonstrating the Continuity of Q as the Point (x, y) Passes Through a Shock.

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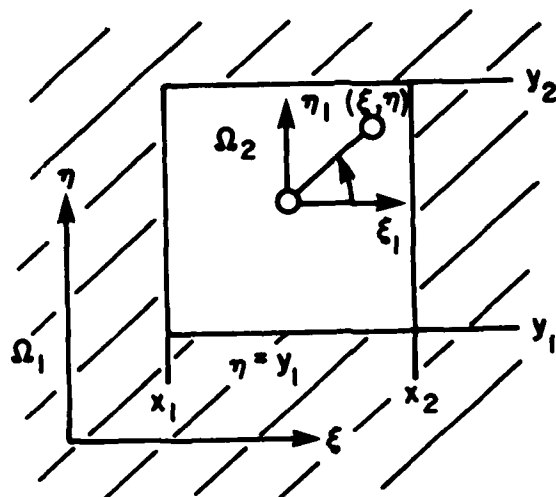


Figure 4. Coordinate Systems Used for the Differentiation of a Double Integral.

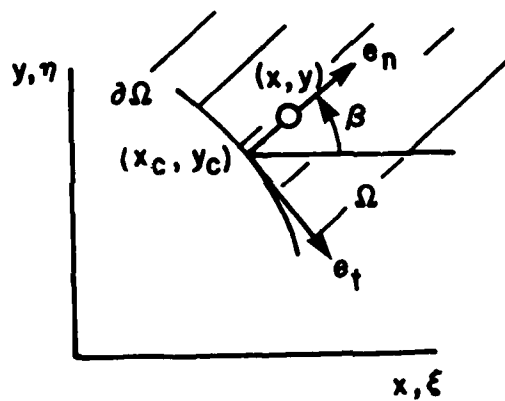


Figure 5. Limiting Case Where a Point (x, y) of the Interior Approaches a Point (x_c, y_c) of the Contour.

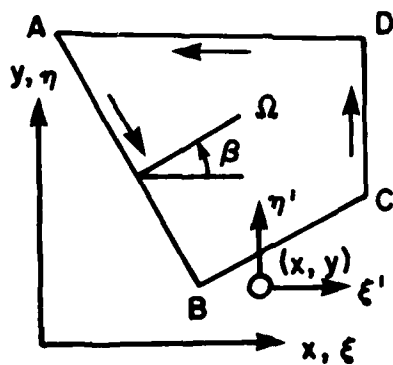


Figure 6. Notation for the Evaluation of a Double Integral.

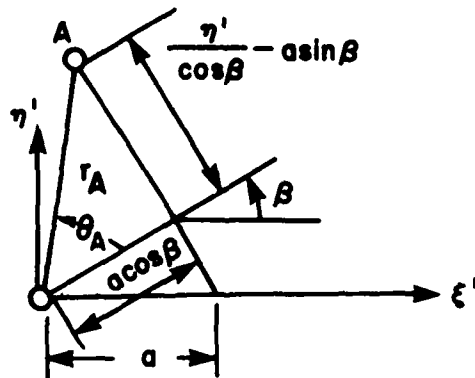


Figure 7. Geometric Interpretation of r_A and θ_A .

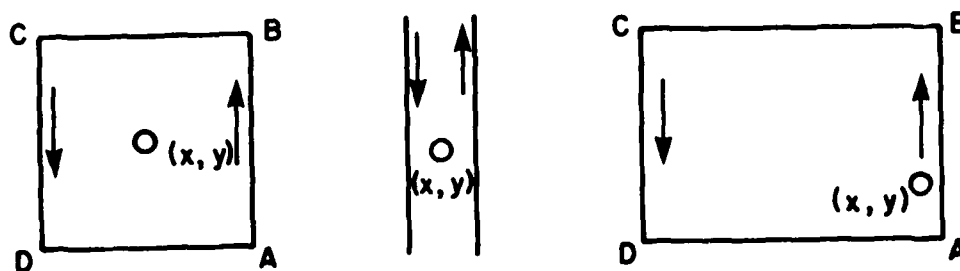


Figure 8. Different Cases in the Evaluation of a Double Integral.

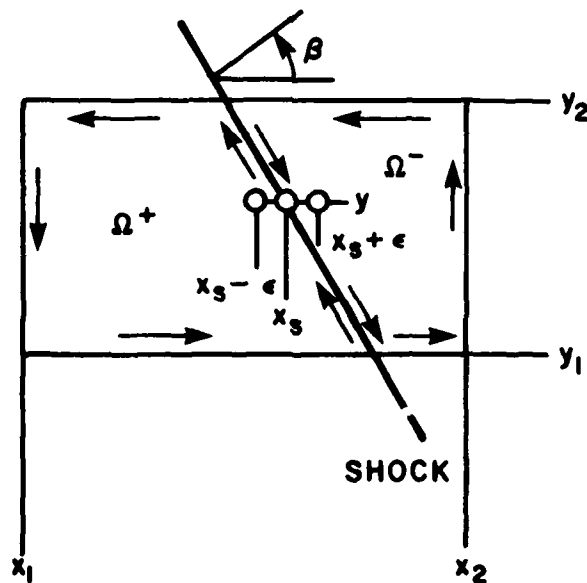


Figure 9. Discussion of Certain Shock Conditions.

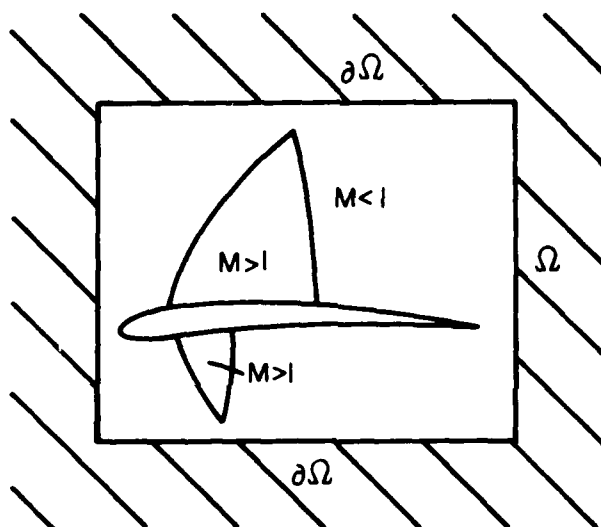


Figure 10. Combination of Finite Difference and Integral Equation Methods.

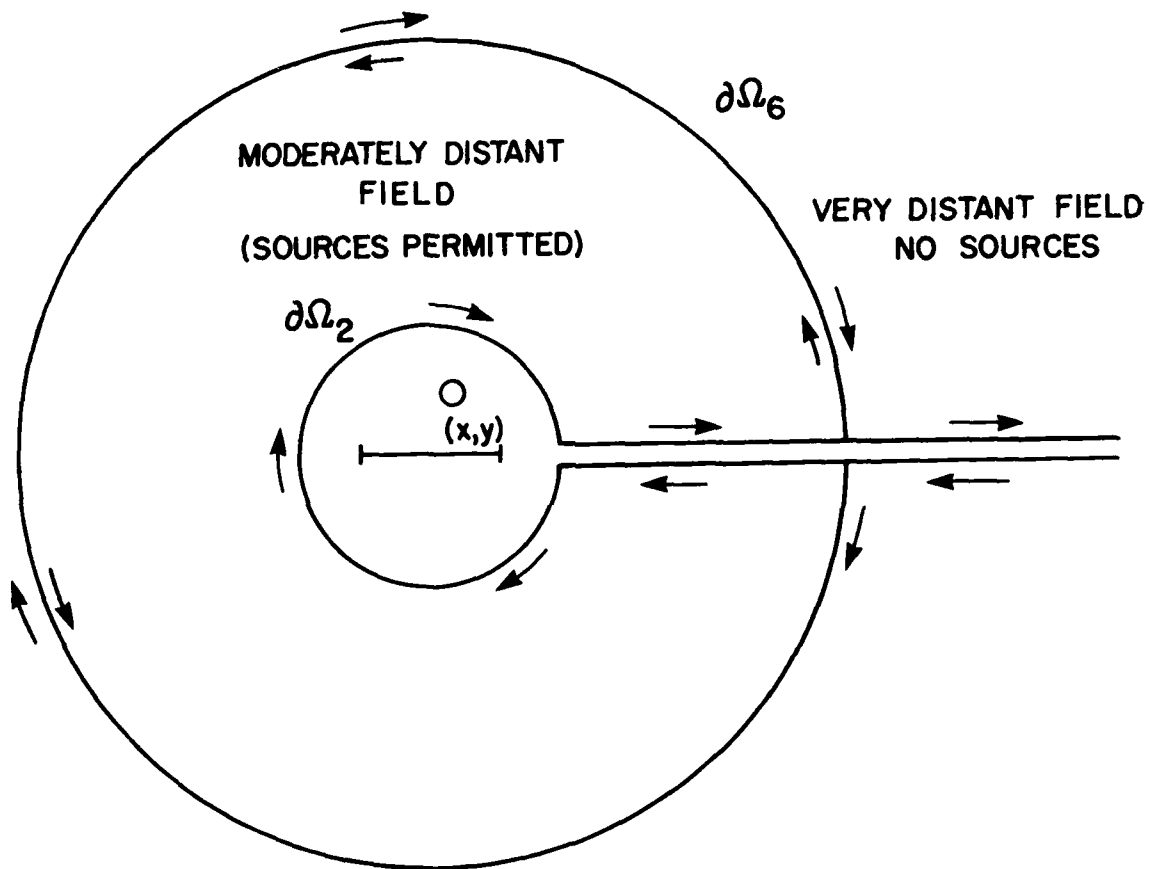


Figure 11. Derivation of Conditions Along $\partial\Omega_2$ if Sources are Present in the Moderately Distant Field.

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